Spherical deconstruction and the Maldacena-Nunez compactification.

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SPHERICAL DECONSTRUCTION AND THE MALDACENA-NÚÑEZ COMPACTIFICATION

By
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To my parents

. . . it is more important to have beauty in one's equations than to have them fit experiment.
Paul Adrien Maurice Dirac, 1963.

Abstract

A comprehensive understanding of four-dimensional confining gauge theories remains an outstanding problem for particle physics. The AdS/CFT correspondence is an invaluable tool that allows a supersymmetric gauge theory at strong coupling to be studied using classical supergravity. The Maldacena-Núñez background has proven successful in studying confining gauge theories using a gravity dual. In the IR, the dual gauge theory is the $\mathcal{N} = 1$ SUSY Yang-Mills theory, however in the UV the Kaluza-Klein modes of a 2-sphere enter the picture and the theory becomes six-dimensional. Unfortunately, these Kaluza-Klein modes are at the same energy scale as the strong coupling dynamics of the gauge theory making it difficult to differentiate between the two effects. This problem would be resolved if the Kaluza-Klein modes were decoupled, which is impossible within the supergravity regime of the gravity dual. Through the application of deconstruction, my research demonstrates that the Higgsed $\mathcal{N} = 1^*$ SUSY Yang-Mills theory is dual to the full string solution of the Maldacena-Núñez background.

In particular, this Thesis calculates the exact classical spectrum of both the Maldacena-Núñez compactified gauge theory and the Higgsed $\mathcal{N} = 1^*$ SUSY Yang-Mills theory, for a $U(1)$ gauge group. In the limit $N \to \infty$, the two spectra are identical and this equivalence generalises to the case of a $U(p)$ gauge group. An explicit comparison of the two classical $U(1)$ actions demonstrates that this equivalence is also present at the level of the classical action. In addition, a web of dualities within the two theories along with this classical equivalence indicates that, in fact, the equivalence is valid at the quantum level. The $\mathcal{N} = 1^*$ SUSY Yang-Mills theory deconstructs the Maldacena-Núñez compactified little string theory.

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Chapter 1

Introduction

Physics aims to provide a comprehensive description of the physical phenomena of nature. The greatest triumph of modern theoretical physics has been the construction of the *standard model of particle physics*, a description of all known particle physics. Extensive experimental studies have verified its predictions and there is no evidence of any violations.[1] The standard model is constructed in the language of *quantum gauge theory*, a framework that has proven extremely successful at describing the quantum behaviour of the fundamental forces, with the notable exception of gravity. In quantum gauge theory, the action of a force is attributed to the exchange of (virtual) gauge bosons. The quarks and leptons are the sources of gauge fields that spread throughout spacetime. Quantum fluctuations of this gauge field can form gauge bosons for a brief period of time due to Heisenberg's uncertainity principle,

$$\Delta E\,\Delta t \geq \hbar \tag{1.1}$$

By continuously exchanging gauge bosons, energy and momentum is transported between particles, transmitting the action of a force.

The standard model describes the action of three fundamental forces, the *electromagnetic* force, the *weak* force and the *strong* force. It is not a single, unified description of these fundamental forces, but a composite theory with three individual

[1]The discovery of massive neutrinos should not be considered a violation of the standard model. The standard model was designed to produce massless neutrinos to fit the experimental evidence of the 1970's.

components. The *electroweak theory* describes the action of the electromagnetic and weak interactions. Electromagnetic interactions are described by *quantum electrodynamics* (QED), a $U(1)$ gauge theory mediated by the massless *photon.* A consistent description of the weak interaction can only acheived when unified with electromagnetic interactions. The electroweak theory is a $SU(2) \times U(1)$ gauge theory, mediated by massless gauge bosons.[2] *Quantum chromodynamics* (QCD) describes the strong interaction, a $SU(3)$ gauge theory mediated by eight massless *gluons.* The final component of the standard model is the Higgs field, a scalar field that is responsible for spontaneously breaking the electroweak gauge group $SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}$.[3] The unbroken $U(1)_{em}$ gauge field gives rise to QED, whilst the remaining three bosons of the weak interaction, $W^{\pm}$ and Z^0, become massive. Mass is dynamically generated in the standard model through interactions between the Higgs boson and the other standard model particles.

The study of particle physics interactions involves the evaluation of various scattering amplitudes. These quantities are calculated by evaluating correlation functions constructed from the path integral, with an action $S[\phi, \dots]$. In general, the correlation functions of an interacting quantum field theory cannot be evaluated exactly. For simplicity, consider a scalar field theory with the Lagrangian [1, 2],

$$\mathcal{L}_b[\phi(x)] = -\frac{1}{2}\,\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}\,m_b^2\,\phi(x)^2 - \frac{1}{4!}\,g_b\,\phi(x)^4 \tag{1.2}$$

To describe the propagation of a particle from x_2 to x_1, the two-point function is evaluated.

$$< 0|\phi(x_1)\phi(x_2)|0 >= \int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{i\int d^4x(\mathcal{L}_b[\phi(x)]+J(x)\phi(x))} \tag{1.3}$$

where $J(x)$ is an external source. For an interacting theory ($g_b \neq 0$) this path integral cannot be evaluated exactly. One approach to solve the path integral is to expand the interacting term of the Lagrangian as a power series and evaluate each term of

[2]Fermi's theory is the independent desciption of the weak interaction but can only be considered an effective theory due to its non-renormalisable nature.

[3]The only part of the standard model that still awaits verification.

the path integral independently.

$$< 0|\phi(x_1)\phi(x_2)|0 > = \int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, e^{i\int d^4x\left(\mathcal{L}_{bf}+J(x)\phi(x)\right)}$$
$$\times \left(1 - \frac{i}{4!}\int d^4y\, g_b\, \phi(y)^4 - \frac{1}{2\,(4!)^2}\int d^4y\, d^4z\, g_b^2\, \phi(y)^4\phi(z)^4 + O(g_b^3)\right) \tag{1.4}$$

where $\mathcal{L}_{bf}$ is the free Lagrangian of (1.2). This approach is conceptualised by the use of *Feynman diagrams* representing the individual terms of the path integral [3]. Unfortunately this series is asymptotic and does not converge, so the path integral cannot be solved exactly using perturbation theory. If the coupling is *weak* ($g_b \ll 1$) then the perturbative series provides an approximation of the path integral, up to $O(1/g_b^2)$, however if the coupling is *strong* ($g_b \gg 1$) then $1/g_b \ll 1$ and the perturbative series is a poor approximation. Provided the coupling is sufficiently weak it appears that perturbation theory would provide a good approximation of the path integral. Unfortunately difficulties are encountered when evaluating Feynman loop diagrams. As particle states within the interior of a Feynman diagram are not restricted by a *mass-shell condition*, all momentum travelling around a loop must be integrated over. It is common for these integrals to be divergent, which would result in a divergent correlation function, which cannot be physical. The divergences can be regulated by imposing a momentum cut-off Λ_{UV} to keep the integrals finite.[4] In fact, it is physically acceptable to have divergent quantities in a physical theory provided they are not observable [1]. The bare mass m_b and bare coupling g_b in the Lagrangian (1.2) are not the physical mass and physical coupling that would be measured in a scattering experiment. The physical parameters such as the mass m and coupling g are functions of the bare parameters and the cut-off Λ_{UV}. The divergences of the Feynman loop diagrams are removed by varying the bare parameters m_b and g_b with the cut-off, whilst keeping the physical parameters m and g fixed [1]. This is *renormalisation.* If the number of parameters that must be reparameterised is finite then the theory can be renormalised, however if the number of parameters is infinite

[4]The use of a momentum cut-off to regulate the theory is simpler on a conceptual level, but it will often break many important symmetries of the theory and in practice a regularisation will be used such that the symmetries of the theory are preserved.

then the theory has no predictive power because any theory can be obtained by an arbitrary reparameterisation and the theory is said to be non-renormalisable. Fermi's theory of the weak interaction is an example of a non-renormalisable quantum field theory.

Renormalisation reparameterises the bare couplings (e.g. g_b) to scale with the cut-off Λ_{UV} in such a way that the physical couplings remain independent of the cut-off [1]. The electroweak theory is found to be weakly coupled at low-energies (IR) and becomes strongly coupled at higher energies (UV). Conversely, QCD is weakly coupled in the UV and strongly coupled in IR [1]. As the cut-off $\Lambda_{UV} \to \infty$, QCD becomes asymptotically free and can be studied at any energy in the UV, it is said to have a continuum limit [4]. QED does not have a continuum limit, as the cut-off $\Lambda_{UV} \to \infty$ the coupling diverges at a finite energy in the UV. This *Landau pole* prevents QED from being studied at an arbitrary high energy scale. QCD flows to a (trivial) UV fixed point in the space of couplings, a massless free field theory. Any theory that flows to a UV fixed point in the space of couplings has a continuum limit [4]. The presence of a Landau pole states that the theory is only valid up to a particular energy scale at which new physics must become manifest [1]. Neither the electroweak theory (due to QED) or scalar field theories (such as the Higgs particle) have a continuum limit [4]. As a consequence the standard model also has no continuum limit, the standard model is not a complete description of particle physics (even if you ignore gravity).

Perturbation theory can be used to study the strong interaction at weak coupling $g \ll 1$, but is unable to study the strong coupling dynamics, such as the structure of a proton. The identification of a non-perturbative description of QCD remains an outstanding problem for particle physics. At strong coupling it is thought that the colour charges of the strong interaction become confined, the force between the charges is directly proportional to their separation so it takes an infinite amount of energy to separate the charges to spatial infinity [2]. The chromoelectric flux forms a string-like structure between the colour charges. This picture of confinement led to the construction of string theory as a theory of the strong interaction. String theory successfully described some qualitative features of confinement, such as Regge trajectories, but contained many unwanted features such as a massless spin-2 particle, an unstable tachyonic ground state and twenty six spacetime dimensions [5]. With the success of QCD, bosonic string theory was abandoned as a description of the strong interaction. Successive attempts to understand confinement in QCD have utilised non-perturbative objects, such as solitons, and exploited various dualities. A duality is two different descriptions of the same physical theory. The most useful dualities link a theory at weak coupling to a theory at strong coupling, allowing the use of perturbative techniques to study strong coupling dynamics.

Whilst the standard model is a successful description of particle physics, due to its lack of a continuum limit it is incapable of describing physics beyond $\sim 10^3$ GeV [6]. Furthermore, quantum gauge theory has been unable to describe gravity, all attempts to quantise *general relativity* result in non-renormalisable quantum field theories. A description of physics beyond the standard model is unknown due to a lack of experimental and observational evidence of physics at these energy scales. When the standard model was being constructed there was a large amount of experimental evidence to guide and motivate the model builders of the time. Without any experimental evidence of the physics beyond the standard model, theorists have resorted to the concepts of enhanced symmetries, unification and mathematical 'beauty' to construct new theories. One proposal is the unification of the standard model interactions, called *grand unification* [3]. Extrapolation of the running couplings of the electroweak theory and QCD suggests that the couplings converge at $\sim 10^{15}$ GeV. This prompted theorists to propose gauge theories with enhanced gauge groups, such as $SU(5)$, which have a $SU(3) \times SU(2) \times U(1)$ subgroup. Grand unified theories have had some success [3] but they do not incorporate gravity and highlight the *hierarchy problem*. If grand unification occurs at $\sim 10^{15}$ GeV, then why are the masses of the standard model particles so small? If a symmetry is spontaneously broken at $\sim 10^{15}$ GeV, then one would expect the massive particles created via the Higgs mechanism would have a mass at the same order. Instead the standard model particles have masses at the same order as the electroweak symmetry breaking scale. The standard model can achieve such small masses through the fine-tuning of the radiative corrections to the Higgs mass (the higher-order Feynman diagrams), however the fine-tuning of radiative corrections is considered unnatural and a more systematic understanding would be desirable. Furthermore, the radiative corrections would be very sensitive to physics beyond the standard model [7], an indication that the electroweak scale is stablised by an unknown theory at energies $> 10^3$ GeV. An extension that unifies the standard model's forces and solves the hierarchy problem is *supersymmetry.* Supersymmetry is a proposed additional symmetry that exists between the bosons and the fermions. As a global symmetry, supersymmetry solves the hierarchy problem: the interactions of the new particles is such that their radiative corrections exactly cancel the radiative corrections of the standard model particles [7]. Invariance under local supersymmetry transformations implies invariance under general coordinate transformations, therefore as a local symmetry, supersymmetry is a theory of gravity called *supergravity.*

The best candidate for a quantum theory of gravity and a fundamental theory of particle physics is *supersymmetric string theory.*[5] Bosonic string theory is invariant under general coordinate transformations and contains a massless spin-2 particle. This lead Schwarz and Scherk [8] to propose that string theory is a theory of gravity. The addition of supersymmetry to bosonic string theory removes the unstable tachyonic ground state and reduces the dimensionality to ten spacetime dimensions. The only consistent string theories are supersymmetric and exist in ten spacetime dimensions. As a fundamental theory, particles in string theory are identified as different vibrational modes of an elementary quantum string [5]. Whilst string theory is a promising candidate for a quantum theory of gravity, so far it has been unable to provide any physical predictions to allow experimental verification [5].

In an attempt to study QCD at strong coupling theorists tried to expand QCD in a dimensionless parameter and study the theory at the lowest order. QCD does not actually contain any dimensionless parameters. The QCD gauge coupling g is

[5]The term string theory usually refers to supersymmetric string theory rather than bosonic string theory.

not a suitable parameter because it is a function of the energy scale [1]. Instead 't Hooft proposed enlarging the gauge group $SU(3) \to SU(N)$. When performing a perturbative expansion, the expansion is parameterised in $1/N$ as well as the QCD coupling g. In the limit $N \to \infty$, 't Hooft noticed that the expansion was similar to an expansion in perturbative string theory and proposed that a connection exists between string theory and large-N gauge theories [9].

D-branes form an important connection between string theory and gauge theory. D-branes are hyperplanes within the ten-dimensional bulk spacetime, defined by the end-points of open strings with Dirichlet boundary conditions [5]. The theory of massless open strings ($\alpha' \to 0$) on the worldvolume of a Dp-brane (with p spatial dimensions) is found to be a $(p+1)$-dimensional supersymmetric gauge theory. A D-brane preserves half of a string theory's thirty two supercharges, so this low-energy supersymmetric gauge theory has sixteen supercharges. Furthermore, D-branes act as sources of closed strings [10, 11]. It was the dual nature of D-branes that lead Maldacena to conjecture that a duality exists between gauge theories and closed string theories [12, 10, 11]. Maldacena studied a set of N D3-branes of Type IIb string theory. The theory of massless open strings on the worldvolume of ND3-branes is the $\mathcal{N} = 4$ SUSY Yang-Mills theory with a $U(N)$ gauge group, on $\Re^{3,1}$. In the closed string theory, the spacetime geometry close to the D3-branes is $AdS_5 \times S^5$. This suggests a duality between the gauge theory on ND3-branes ($\mathcal{N} = 4$ SUSY Yang-Mills theory) and the description of Type IIb closed string theory on $AdS_5 \times S^5$. Maldacena's duality is a weak-strong duality, the strong coupling regime of $\mathcal{N} = 4$ SUSY Yang-Mills is dual to the weak coupling regime of Type IIb closed string theory, and vice versa. Whilst Maldacena's conjecture has not been proved conclusively, it has passed many non-trivial tests. These tests have mostly been performed at low-energy where $\alpha' \to 0$. In this limit the closed string theory reduces to Type IIb supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SUSY Yang-Mills is in the large 't Hooft limit $N \to \infty$, $\lambda \to \infty$ [13]. The importance of Maldacena's duality is the ability to study $\mathcal{N} = 4$ SUSY Yang-Mills theory at strong coupling using classical Type IIb supergravity on $AdS_5 \times S^5$. Maldacena's duality provides an explicit realisation of 't Hooft's conjecture and realises the original objective of string theory: to describe gauge theories at strong coupling using effective superstrings with a string tension $\sqrt{T} = 250$ MeV.

Unfortunately, $\mathcal{N} = 4$ SUSY Yang-Mills does not resemble QCD in any way [10, 11]. The $\mathcal{N} = 4$ theory has maximal supersymmetry for a renormalisable gauge theory and is invariant under conformal transformations. QCD is a confining gauge theory, not conformal and does not possess supersymmetry. In order to study QCD at strong coupling using a gravity dual of the type found by Maldacena, both the conformal invariance and supersymmetry must be broken [10, 11]. Extensions to Maldacena's *AdS/CFT correspondence* have been the subject of many research papers. This Thesis concerns one of the first extensions of the AdS/CFT correspondence that constructs a gravity dual of a $\mathcal{N} = 1$ SUSY Yang-Mills theory. Physicists have studied $\mathcal{N} = 1$ SUSY Yang-Mills theory extensively due to its similarities to QCD, it is a confining gauge theory, possesses chiral symmetry breaking and many other features. Maldacena and Núñez constructed a $\mathcal{N} = 1$ SUSY Yang-Mills theory on $\Re^{3,1}$ from a *little string theory* (LST) [14]. LST is the non-trivial, interacting theory on the worldvolume of $p > 1$ coincident NS5-branes of Type IIb supergravity, in the limit where the string coupling $g_s \to 0$ [15].[6] In this limit, the worldvolume theory of the NS5-branes decouples from the closed strings of the bulk spacetime. The resulting theory is an interacting, non-gravitational theory with string-like excitations. At low-energy ($\alpha' \to 0$), LST becomes $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory on $\Re^{5,1}$ [15]. To reduce the dimensionality of the dual gauge theory, Maldacena and Núñez wrapped the NS5-branes on a non-contractable 2-cycle of a Calabi-Yau three-fold (CY_3) [14]. A Calabi-Yau manifold is a generalisation of a 2-torus, with the 2-sphere of the six-dimensional CY_3 being analogous to a circle on a 2-torus. As will be demonstrated in Section 3.5, a conventional compactification on a 2-sphere breaks all supersymmetries. Maldacena and Núñez preserved a quarter of the NS5-brane's supersymmetries by embedding the spin connection of the 2-sphere in the R-symmetry of the theory [14]. The twisted compactification preserves four supersymmetries of the original thirty two.[7] In the IR the Maldacena-Núñez compactified gauge theory

[6] At least two NS5-branes are required for a non-trivial $g_s \to 0$ limit.

[7] A NS5-brane is related to a D5-brane via S-duality, $g_s \to 1/g_s$. A D-brane preserves half of the thirty two supersymmetries of string theory, whilst the twisted compactification of a D5-brane

is a $\mathcal{N} = 1$ SUSY Yang-Mills theory on $\Re^{3,1}$, whilst in the UV the theory becomes LST on $\Re^{3,1} \times S^2$. As the energy scale μ increases, the particles start to probe the additional two dimensions of the 2-sphere. If $\alpha' \ll R$ (radius of the 2-sphere), then the gauge theory will become a $\mathcal{N} = 1$ SUSY Yang-Mills theory on $\Re^{3,1} \times S^2$ at $E \sim 1/R$, before the stringy excitations become manifest at $E \sim 1/\sqrt{\alpha'}$.

IR	$\longrightarrow$	UV
$\mathcal{N} = 1$ SUSY YM on $\Re^{3,1}$	$\mathcal{N} = 1$ SUSY YM on $\Re^{3,1} \times S^2$	LST on $\Re^{3,1} \times S^2$

The Maldacena-Núñez compactification of a gauge theory has proven very successful at studying aspects of four-dimensional confining gauge theories using the dual gravity theory, a Type IIb supergravity in a geometry which is topologically $\Re^{3,1} \times \Re \times S^2 \times S^3$ [16] and refered to as the Maldacena-Núñez background. Unfortunately, the gravity theory is not dual to the four-dimensional $\mathcal{N} = 1$ SUSY Yang-Mills theory, it is dual to a six-dimensional SUSY gauge theory. The four-dimensional gauge theory is contaminated by the Kaluza-Klein modes of the 2-sphere [16]. If these Kaluza-Klein modes were decoupled, the Maldacena-Núñez background would be dual to the four-dimensional $\mathcal{N} = 1$ SUSY Yang-Mills theory. It is not possible to decouple the Kaluza-Klein modes within the supergravity regime of the Maldacena-Núñez background, this is the *decoupling problem* [16]. The Kaluza-Klein modes have a mass (squared) $M_{kk}^2 = 1/(\mathrm{Vol}\, S^2)$. To decouple the Kaluza-Klein modes, the radius of the 2-sphere in the dual gravity geometry must be taken to zero. Within the supergravity regime, the radius of the 2-sphere can only be reduced to a finite size R_0. Furthermore, at this finite radius the Kaluza-Klein modes have a mass $M_{kk}^2 \sim \Lambda_{\mathcal{N}=1}^2$, where $\Lambda_{\mathcal{N}=1}$ is the energy at which the $\mathcal{N} = 1$ theory becomes strongly coupled [16]. When using the supergravity dual to understand the four-dimensional gauge theory, the decoupling problem causes difficulties in trying to differentiate between the strong coupling and the Kaluza-Klein dynamics. In order to

preserves a further quarter, hence the gauge theory on the spherically wrapped NS5/D5-branes has four supersymmetries.

take the radius $R \to 0$ and decouple the Kaluza-Klein modes, the full string solution to the Maldacena-Núñez background must be used, a solution which is unknown [16].

This Thesis will use *deconstruction* [17, 18, 19] to help identify the dual four-dimensional gauge theory. Higher-dimensional gauge theories (dimension greater than four) are non-renormalisable so a perturbative definition is useless. To study higher-dimensional field theories a non-perturbative definition (a UV completion) is required. The UV completion of a six-dimensional gauge theory is LST [20]. The existence of LST can be proven by string theory, but it does not yield a Lagrangian description of the theory. Deconstruction is a technique that identifies the Kaluza-Klein modes of a higher-dimensional field theory as the massive states of a spontaneously broken four-dimensional gauge theory. In deconstruction it is found that the Higgs phase of the four-dimensional theory can be re-interpreted as a field theory with addition compact, discretised dimensions [17, 20]. The discretised nature of the extra dimensions provides the deconstructed theory with a natural UV cut-off. Ideally, the full Lorentz invariance of the higher-dimensional field theory is restored in an appropriate limit. With deconstruction, a higher-dimensional field theory can be studied as a special limit of a four-dimensional field theory [20]. Deconstruction can be used to define the Maldacena-Núñez compactification of LST (and the gauge theory) as a limit of a four-dimensional gauge theory.[8] The candidate four-dimensional theory to deconstruct the Maldacena-Núñez compactification is the $\mathcal{N} = 1^*$ SUSY Yang-Mills theory with a $U(N)$ or $SU(N)$ gauge group. The $\mathcal{N} = 1^*$ theory is a relevent deformation of the $\mathcal{N} = 4$ theory where the chiral multiplets become massive [22]. It comprises of a $U(N)$ (or $SU(N)$) vector multiplet and three massive adjoint chiral multiplets of $\mathcal{N} = 1$ supersymmetry. Each chiral multiplet has a mass m_i, $i = 1, 2, 3$, which for simplicity are set to be equal $m_1 = m_2 = m_3 = \eta$. Schematically, the superpotential of the $\mathcal{N} = 1^*$ theory is,

$$\mathcal{W}(\Phi) = \mathrm{Tr}\left(i\Phi_1[\Phi_2, \Phi_3] + \frac{1}{2}\sum_{i=1}^{3}\Phi_i^2\right) \tag{1.5}$$

[8]See [20] for the deconstruction of a torodially compactified LST and the associated gauge theory.

The F-flatness condition of the $\mathcal{N} = 1^*$ theory is found to coincide with the $SU(2)$ Lie algebra.

$$[\Phi_i, \Phi_j] = i\varepsilon_{ijk}\Phi_k \tag{1.6}$$

The vacuum can be solved by any d-dimensional representation of the $SU(2)$ generators [21, 22]. The Higgs branches correspond to the choice of vacua $< \Phi_i >= \mathbb{1}_p \otimes J_i^{(q)}$, where $J_i^{(q)}$ is the q-dimensional irreducible representation of the $SU(2)$ generators. This choice of vacuum breaks the gauge group $U(N = pq) \to U(p)$ $\big($or $SU(N) \to SU(p)\big)$ via the Higgs mechanism. The extra dimensions of deconstruction emerge via the mechanism seen in M(atrix) theory [23]. The vacua of the complex scalars forms a fuzzy sphere, a discrete non-commutative version of the 2-sphere [24]. M(atrix) theory identifies a correspondence between matrices on the fuzzy sphere and fields on a non-commutative 2-sphere S^2_n. By expanding the $\mathcal{N} = 1^*$ theory about this vacuum, the theory becomes a six-dimensional non-commutative supersymmetric gauge theory. The non-commutative nature of the extra dimensions provides a natural UV cut-off for the six-dimensional non-commutative field theory. Classically, in the limit $N \to \infty$ the theory becomes a commutative continuum theory on $\Re^{3,1} \times S^2$.

The appearance of extra dimensions can be seen in the dual gravity description of the $\mathcal{N} = 1^*$ theory. The relevent deformation of the $\mathcal{N} = 4$ theory corresponds to the presence of a non-trivial RR 3-form flux [22]. The flux acts like a magnetic field on the ND3-branes of the $\mathcal{N} = 4$ SUSY Yang-Mills theory producing a version of the Myers effect. In analogy to a dipole moving through a magnetic field or a dielectric in an electric field, the ND3-branes are distributed on a 2-sphere (forming a discrete subspace) in the six dimensions transverse to the D3-branes. The complex scalars of the $\mathcal{N} = 4$ theory Φ_i, label the complex coordinates of the six-dimensional bulk spacetime transverse to the D3-branes. The eigenvalues of the complex scalars Φ_i denote the position of each D3-brane in the transverse spacetime. The complex scalars satisfy the $SU(2)$ Lie algebra and in the Higgs phase, the D3-branes lie on a fuzzy sphere, Figure 1.1. The Hodge dual of the 3-form flux F_3 is ${}^*F_3 = F_7$, which couples to a D5-brane. The N units of D3-brane charge become N units of magnetic flux through the 2-sphere, leading to a non-commutative worldvolume [25, 19]. In the Higgs vacuum, the ND3-branes have become spherical polarised into p spherically wrapped D5-branes. The Maldacena-Núñez background is also a theory of spherically

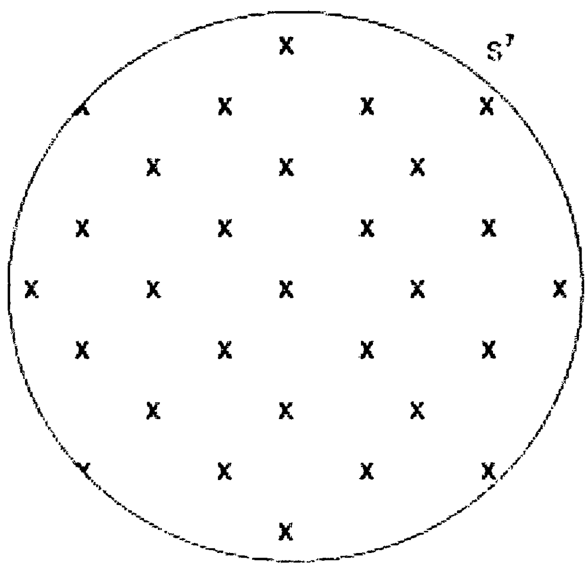


Figure 1.1: D3-branes forming a fuzzy sphere within the six-dimensional space transverse the D3-branes

wrapped D5-branes. The research presented in this Thesis will identify a connection between these two theories: the theory of D5-branes wrapped on a non-trivial two-cycle of a CY_3 that was considered by Maldacena and Núñez and the theory of D5-branes wrapped on a trivial 2-cycle in the presence of external flux. A comparison of the classical spectrum of the Higgsed $\mathcal{N} = 1^*$ theory with the classical Kaluza-Klein spectrum of the Maldacena-Núñez compactified gauge theory shows them to be exactly equivalent in the limit $N \to \infty$ ($q \to \infty$ with $p =$ fixed). This equivalence is also valid at finite N provided the Kaluza-Klein spectrum of the Maldacena-Núñez compactified gauge theory is truncated appropriately. Furthermore, it will be shown that this equivalence holds at the level of the classical action.[9]

The classical equivalence of the Maldacena-Núñez compactified gauge theory and the Higgsed $\mathcal{N} = 1^*$ theory demonstrates that, in the limit $N \to \infty$, they are two different descriptions of the same classical theory. The theory has a scale dictated by the radius of the 2-sphere R, which is inversely proportional to the mass parameter of the $\mathcal{N} = 1^*$ theory, $R \sim \eta^{-1}$. In the limit $N \to \infty$, the theory has two regimes, one at distances $L > R$ and another at distances $L < R$. At distances $L > R$, the fields have insufficient energy to probe the 2-sphere, the theory is a four-dimensional $\mathcal{N} = 1^*$ SUSY Yang-Mills theory with a $U(p)$ (or $SU(p)$) gauge group with coupling $g_4^2 = g_{ym}^2/q$, where g_{ym} is the Yang-Mills coupling constant of the $U(N)$ (or $SU(N)$) $\mathcal{N} = 1^*$ SUSY Yang-Mills theory. At energies $\mu \ll \eta$, the $\mathcal{N} = 1^*$ theory reduces to $\mathcal{N} = 1$ SUSY Yang-Mills theory. This is the four-dimensional theory that the Maldacena-Núñez compactified gauge theory aimed to study via the supergravity dual and is subject to the decoupling problem. The Kaluza-Klein modes of the 2-sphere (in the dual gauge theory) have a mass (squared) $\Lambda_{kk}^2 \sim \eta^2$, which is at the same order as the strong coupling scale of the $\mathcal{N} = 1$ SUSY Yang-Mills theory $\Lambda_{\mathcal{N}=1} \sim \Lambda_{KK}$. From the 1-loop β-function of the $\mathcal{N} = 1$ SUSY Yang-Mills theory [26],

$$\frac{8\pi^2}{g^2(\mu)} = \frac{3}{2} T(adj) \ln\left(\frac{\mu}{\Lambda_{\mathcal{N}=1}}\right) \tag{1.7}$$

$\frac{1}{2} T(adj) = p$ for a $SU(p)$ gauge group [26, 2].

$$\Lambda_{\mathcal{N}=1} = \mu \exp\left(-\frac{8\pi^2}{3p\, g^2(\mu)}\right) \tag{1.8}$$

This expression is valid for any energy scale μ at weak coupling. The energy scale can be taken as the momentum cut-off of the $\mathcal{N} = 1$ SUSY Yang-Mills theory $\mu = \Lambda_{UV}$. If the momentum cut-off is given by the mass parameter of the $\mathcal{N} = 1^*$ theory.

$$\Lambda_{UV} \sim \eta \qquad \text{and} \qquad g^2(\mu) = \frac{g_{ym}^2}{q} \tag{1.9}$$

[9] A comparison between the classical spectra and actions will be performed for the case of an abelian gauge group. The comparison of the spectra has a trivial extension in the case of a non-abelian gauge group.

The $\mathcal{N} = 1$ SUSY Yang-Mills theory becomes strong coupling at,

$$\Lambda_{\mathcal{N}=1} = \eta \exp\left(-\frac{8\pi^2}{3p}\frac{q}{g_{ym}^2}\right) = \eta \exp\left(-\frac{8\pi^2}{3p}\frac{4\pi R^2}{g_6^2}\right) \tag{1.10}$$

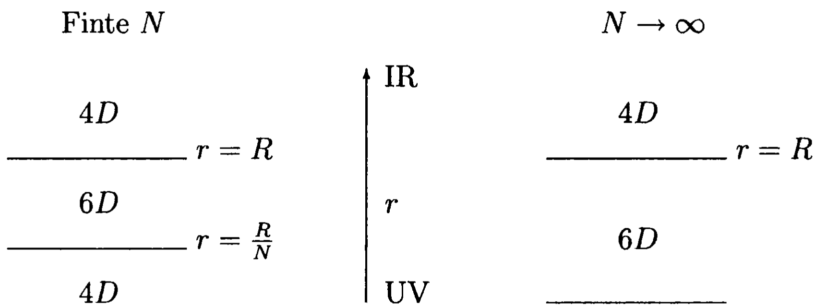
The decoupling problem is resolved if $\Lambda_{\mathcal{N}=1} \ll \Lambda_{KK}$. The Kaluza-Klein modes will then be at a higher energy than the strong coupling regime, i.e.

$$\frac{\Lambda_{\mathcal{N}=1}}{\eta} = \exp\left(-\frac{8\pi^2}{3p}\frac{4\pi R^2}{g_6^2}\right) \ll 1 \tag{1.11}$$

This limit is satisfied by $g_6^2 \ll 4\pi R^2$. By making the 2-sphere of the dual gauge theory sufficiently large, the Kaluza-Klein modes are taken to an energy $\Lambda_{KK} \gg \Lambda_{\mathcal{N}=1}$.[10] At distances $L < R$ the fields have sufficient energy to probe the 2-sphere and the theory is a six-dimensional, $U(p)$ (or $SU(p)$) supersymmetric gauge theory with coupling $g_6^2 = 4\pi R^2 g_{ym}^2/q$. From a four-dimensional viewpoint the theory is a $\mathcal{N} = 1$ SUSY Yang-Mills theory with the massive Kaluza-Klein modes representing the degrees of freedom on the 2-sphere. At finite N, the Kaluza-Klein spectrum of this regime is truncated and the 2-sphere is non-commutative, producing a non-commutative field theory. Furthermore, when N is finite there is an addition regime at distances $L < R/q$, the theory becomes four-dimensional again and the full $U(N)$ (or $SU(N)$) gauge symmetry is restored. In this regime, the effective UV cut-off or lattice spacing is $a = R/q$. This behaviour is typical of deconstruction.

The classical equivalence between the Higgsed $\mathcal{N} = 1^*$ theory (in the limit $N \to \infty$) and the Maldacena-Núñez compactified gauge theory will be demonstrated in Chapters 4 and 6. The classical nature of this equivalence restricts the result to weak-coupling. It is important to ask whether the four-dimensional Higgsed $\mathcal{N} = 1^*$ theory can consistently define a continuum limit for an interacting six-dimensional non-commutative gauge theory when the 'lattice spacing' of the non-commutative 2-sphere $a \to 0$. For $N = pq$, $a = R/q$, so the limit $a \to 0$ corresponds to the naive continuum limit $q \to \infty$. To identify an interacting six-dimensional gauge theory, the limit $q \to \infty$ must be taken such that g_6^2 = fixed, g_4^2 = fixed and (for simplicity) R = fixed. In this limit it is found that as $q \to \infty$ (p = fixed), $g_{ym}^2 \to \infty$, the

[10]This 2-sphere is the compact manifold of the dual gauge theory and does not refer to the 2-sphere of the dual gravity theory.



theory becomes strongly coupled and the classical analysis/equivalence outline above and describe in detail in Chapters 4 and 6 is invalid. This is a generic feature of deconstruction.

The classical analysis presented in this Thesis is incapable of identifying the full continuum limit. Through the dualities present within the $\mathcal{N} = 1^*$ theory and the Maldacena-Núñez background, there is evidence that the identity of the continuum limit is the Maldacena-Núñez compactified LST. The classical equivalence between the $\mathcal{N} = 1^*$ theory and the Maldacena-Núñez compactified gauge theory identifies a classical equivalence between p D5-branes spherically wrapped on a trivial 2-cycle in the presence of external flux and p D5-branes spherically wrapped on a non-trivial 2-cycle of a CY_3. The continuum limit corresponds to $q \to \infty$ for $R =$ fixed and $g_{ym}^2/q =$ fixed, i.e. $g_{ym}^2 \to \infty$. In this limit the $\mathcal{N} = 1^*$ theory is strongly coupled and confining. The Higgs phase of the $\mathcal{N} = 1^*$ theory is related to the confining phase via S-duality [22].

$$g_{ym}^2 \longrightarrow \tilde{g}_{ym}^2 = \frac{16\pi^2}{g_{ym}^2} \tag{1.12}$$

In the gravity dual of the $\mathcal{N} = 1^*$ theory, S-duality relates the p spherically wrapped D5-branes to p spherically wrapped NS5-branes [22].

$$g_s \longrightarrow \tilde{g}_s = \frac{1}{g_s} \tag{1.13}$$

The relationship between the string coupling on the D-branes g_s and the Yang-Mills coupling of the $\mathcal{N} = 1^*$ theory in the Higgs phase is $g_s = 4\pi g_{ym}^2$ [22]. By S-duality, this same relationship between the NS5-branes and the confining $\mathcal{N} = 1^*$ theory is $\tilde{g}_s = 4\pi\tilde{g}_{ym}^2$. The continuum limit now corresponds to $q \to \infty$ with $\tilde{g}_{ym}^2 q$ = fixed. As $q \to \infty$, $\tilde{g}_s \to 0$. This is precisely the limit for LST. The continuum theory is p spherically wrapped coincident NS5-branes in the limit $\tilde{g}_s \to 0$, which is a LST on $\Re^{3,1} \times S^2$. The little strings are the chromoelectric flux tubes of the confining $\mathcal{N} = 1^*$ theory. In the Maldacena-Núñez background, S-duality relates the spherically wrapped D5-branes to the spherically wrapped NS5-branes, which in the limit $g_s \to 0$ becomes a LST. Together with the classical equaivalence between the two sets of spherically wrapped D5-branes, this identifies the LST from the $\mathcal{N} = 1^*$ theory as the LST considered by Maldacena and Núñez in [14]. The Higgsed $\mathcal{N} = 1^*$ SUSY Yang-Mills theory is the four-dimensional gauge theory dual to the full Maldacena-Núñez background.[11]

The outline of this Thesis is as follows. Chapter 2 reviews the construction of four-dimensional supersymmetric field theories and ends with the construction of the Lagrangian for the $\mathcal{N} = 1^*$ SUSY Yang-Mills theory. This Chapter is based on a number of textbooks, lecture notes and review papers, primarily [3, 27, 28, 29, 30].

Chapter 3 will discuss higher-dimensional field theories and their compactifications. The discussion of higher-dimensional field theories, specifically spinor fields in diverse dimensions, is based on [3, 31]. Compactifications and the dimensional reduction of higher-dimensional field theories is based on [3, 31, 32], whilst the discussion of field theory in curved spacetime is based on the textbooks [33, 34]. The most important part of this Chapter outlines the mathematics necessary to compactify a field theory on a 2-sphere. It was constructed using several sources [35, 37, 38, 39].

Chapter 4 will discuss the Maldacena-Núñez compactification. It begins with a review of the Maldacena-Núñez compactification from a group theory perspective based on the original work of Maldacena and Núñez [14] and the review papers [10, 11]. It will then construct the bosonic action of the gauge theory and calculate the classical

[11]LST only arises in the limit $g_s \to 0$ of p coincident NS5-branes if $p > 1$. As a result, the connection between the Higgsed $\mathcal{N} = 1^*$ theory and the Maldacena-Núñez compactified LST can only be demonstrated if the classical equivalence is between non-abelian gauge theories. This essential equivalence is observed when the classical spectra are compared directly.

Kaluza-Klein spectrum. I first presented this original research in [40].

Chapter 5 discusses the technique of deconstruction and demonstrates the emergence of extra dimensions in the $\mathcal{N} = 1^*$ SUSY Yang-Mills theory. Starting with the observations of the $\mathcal{N} = 1^*$ SUSY Yang-Mills vacua [21, 22], spatial dimensions are shown to emerge via M(atrix) theory [23]. Chapter 5 ends with the construction of the Higgsed $\mathcal{N} = 1^*$ SUSY Yang-Mills theory by imposing the choice of Higgs vacuum to the Lagrangian of Section 2.6.

Chapter 6 will explicitly apply the deconstruction technique of Chapter 5 to the Higgsed $\mathcal{N} = 1^*$ SUSY Yang-Mills theory. It begins by calculating the classical spectrum and then constructs the effective six-dimensional action. The spectrum and action will be compared to the spectrum and action of the Maldacena-Núñez compactified gauge theory of Chapter 4. I first presented this original research in [40]. Both classical spectra will be calculated explicitly for the case of a $U(1)$ gauge group, each with a trivial extension to a non-abelian unitary gauge group.

Finally, Chapter 7 will present concluding remarks. There are two appendices, Appendix A outlines the conventions and spinor identities used in this Thesis, whilst Appendix B outlines the relationship between the basis vectors of a vector space.

Chapter 2

Supersymmetry

The standard model is not a fundamental theory. As mentioned in the Introduction, supersymmetry [3, 27, 28, 29, 30] was proposed as a possible extension to the standard model. In fact, supersymmetry is the only possible extension of the symmetries in the standard model, within the framework of quantum field theory [30]. Supersymmetry is a generalisation of the Poincaré group to incorporate an internal symmetry group relating bosons and fermions. It was shown by Coleman and Mandula [41], under very general assumptions, that no Lie group can be found which contains the Poincaré group and an internal symmetry group in a non-trivial manner. Whilst there exists such Lie groups, the S-matrix for all processes in such a field theory are one and therefore not physically interesting [30]. Supersymmetry avoids the restriction of the Coleman-Mandula theorem by generalising the concept of a Lie algebra to include anticommutators in addition to the usual commutators of a Lie algebra [30]. These algebras are called superalgebras or graded Lie algebras. Supersymmetry is of great interest to physicists even though there is no experimental evidence to suggest supersymmetry is present in nature. Supersymmetry provides a solution to the hierarchy problem of the standard model and is an important component of string theory, the most promising candidate for a quantum theory of gravity.

This Chapter begins by introducing the supersymmetry algebra and the $\mathcal{N} = 1$ representations in Section 2.1. It then introduces supersymmetric field theory in Section 2.2 before presenting superfields as a tool for constructing $\mathcal{N} = 1$ supersymmetric field theories in Section 2.3. Section 2.4 will use superfields to construct the most general $\mathcal{N}=1$ supersymmetric Lagrangian and discuss the supersymmetric vacua of supersymmetric theories. Section 2.5 will briefly introduce extended supersymmetry and supergravity before constructing the Lagrangian of the $\mathcal{N}=1^*$ SUSY Yang-Mills theory in Section 2.6.

This Chapter only concerns supersymmetry in four spacetime dimensions. An extension to other dimensions will be presented in Section 3.1. The material presented in this Chapter is based on several textbooks and papers, primarily the texts [27, 28, 29, 30, 3], and uses the conventions and notations of Wess and Bagger [27]. Appendix A provides some reference for spinors of $SO(3,1)$.

2.1 Supersymmetry Algebra

Supersymmetry is a $\mathbb{Z}_2$-graded Lie algebra [30]. There are two types of generator in this superalgebra, odd (fermionic) generators and even (bosonic) generators. Schematically, the generators have the following (graded) Lie algebra.

$$\begin{aligned}
[\text{even}, \text{even}] &= \text{even} \\
\{\text{odd}, \text{odd}\} &= \text{even} \\
[\text{even}, \text{odd}] &= \text{odd}
\end{aligned}$$

Square brackets are the usual commutators and the braces are the usual anticommutators. The generalised Jacobi identity is [27],

$$\{A, \{B, C]] \pm \{C, \{A, B]] \pm \{B, \{C, A]] = 0 \tag{2.1.1}$$

The notation $\{\,,\,]$ denotes a commutator or anticommutator based on whether A, B and C are even or odd. The sign is defined by the odd generators, positive if they are cyclic permutations of the first term otherwise they are negative. Supersymmetry is a generalisation of the Poincaré group, which is found to form a subalgebra of the superalgebra [28, 29].

$$\begin{aligned} \left[M^{\mu\nu}, M^{\lambda\rho}\right] &= \eta^{\mu\lambda}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\lambda} - \eta^{\nu\lambda}M^{\mu\rho} + \eta^{\nu\rho}M^{\mu\lambda} & (2.1.2a) \\ \left[M^{\mu\nu}, P^{\lambda}\right] &= -\left(\eta^{\nu\lambda}P^{\mu} - \eta^{\mu\lambda}P^{\nu}\right) & (2.1.2b) \\ \left[P^{\mu}, P^{\nu}\right] &= 0 & (2.1.2c) \end{aligned}$$

The fermionic generators are the supercharges, ${Q_\alpha}^A$ and $\bar{Q}_{\dot\beta B}$. These generators are spinors under the Lorentz group, with the usual spinor indices α and $\dot\beta$ of $SU(2) \times SU(2) \sim SO(3,1)$. They are also representations of an internal symmetry group, these representations being labelled by the indices A and B ($A, B = 1, \ldots, \mathcal{N}$). The barred and unbarred supercharges are charge conjugates of $SO(3,1)$, $\bar{Q}_{\dot\alpha A} = ({Q_\alpha}^A)^\dagger$. The internal symmetry indices are spectators in commutation relations with the Poincaré generators [28]. The commutation relations between the Poincaré generators and the supercharges are as follows [28, 30].

$$\begin{aligned} \left[P^{\mu}, {Q_\alpha}^A\right] &= \left[P^{\mu}, \bar{Q}_{\dot\alpha A}\right] = 0 & (2.1.3a) \\ \left[M^{\mu\nu}, {Q_\alpha}^A\right] &= -{(\sigma^{\mu\nu})_\alpha}^\beta {Q_\beta}^A & (2.1.3b) \\ \left[M^{\mu\nu}, {\bar{Q}^{\dot\alpha}}{}_A\right] &= -{(\bar\sigma^{\mu\nu})^{\dot\alpha}}_{\dot\beta} {\bar{Q}^{\dot\beta}}{}_A & (2.1.3c) \end{aligned}$$

The objects ${(\sigma^{\mu\nu})_\alpha}^\beta$ and ${(\bar\sigma^{\mu\nu})^{\dot\alpha}}_{\dot\beta}$ are spin-$\frac{1}{2}$ representations of the Lorentz group generators, see Appendix A.

The internal symmetry is a Lie group with generators B^r and the Lie algebra,

$$[B^r, B^s] = ic^{rst}B^t \qquad (2.1.4)$$

The objects c^{rst} are structure constants of the algebra. A representation of the generators ${(b^r)^A}_B$ will satisfy,

$$[b^r, b^s] = ic^{rst}b^t \qquad (2.1.5)$$

The commutation relations between the internal symmetry generators B^r and the supercharges are,

$$\begin{aligned} \left[B^r, {Q_\alpha}^A\right] &= -{(b^r)^A}_B {Q_\alpha}^B & (2.1.6a) \\ \left[B^r, \bar{Q}_{\dot\alpha A}\right] &= \bar{Q}_{\dot\alpha B}{(b^r)^B}_A & (2.1.6b) \end{aligned}$$

The supercharges are fermionic generators and satisfy anticommutation relations of the superalgebra.

$$\{Q^A{}_\alpha, \bar{Q}_{\dot\beta B}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \delta^A{}_B \qquad (2.1.7a)$$

$$\{Q_\alpha{}^A, Q_\beta{}^B\} = \varepsilon_{\alpha\beta} Z^{AB} \qquad (2.1.7b)$$

$$\{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\} = \varepsilon_{\dot\alpha\dot\beta} Z^*_{AB} \qquad (2.1.7c)$$

The objects $Z^{AB} = (q^r)^{AB} B^r$, known as central charges, are a linear combination of the internal symmetry generators with complex coefficients $(q^r)^{AB}$. The central charges commute with all the generators of the superalgebra and form an abelian subgroup of the internal (R-)symmetry group [3]. In the absence of central charges the internal symmetry group is $U(\mathcal{N})$. The superalgebra is invariant under the transformation,

$$Q_\alpha{}^A \to U^A{}_B Q_\alpha{}^B \qquad \bar{Q}_{\dot\alpha A} \to \bar{Q}_{\dot\alpha B} U^{\dagger B}{}_A \qquad (2.1.8)$$

for the unitary matrix $U^A{}_B$ of $U(\mathcal{N})$. The presence of central charges reduces this symmetry. The central charges are antisymmetric $Z_{AB} = -Z_{BA}$, so there are no central charges in a $\mathcal{N} = 1$ theory and the internal group is $U(1)$.

Representations of the supersymmetry algebra are constructed in the usual way: by defining fermionic creation and annihilation operators. Excitations are constructed by the successive application of the creation operators on the Clifford vacuum state $|\Omega >$ [27]. Due to the Grassmann nature of these operators the number of excited states is finite. Each representation of the algebra contains the same number of bosons and fermions. The states are characterised by the Casmir operators, $P^2 = P_\mu P^\mu$ and $C^2 = C_{\mu\nu} C^{\mu\nu}$ [29], where P_μ is the 4-momentum and,

$$\begin{aligned} C_{\mu\nu} &= B_\mu P_\nu - B_\nu P_\mu \\ B_\mu &= W_\mu - \frac{1}{4} \bar{Q}_{\dot\alpha} (\bar\sigma_\mu)^{\dot\alpha\alpha} Q_\alpha \end{aligned}$$

where the Pauli-Ljubanskí vector $W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} P^\nu M^{\sigma\rho}$ for massive states and $W_\mu = \lambda P_\mu$ for massless states (λ is helicity). Each state of a distinct irreducible representation of the supersymmetry algebra has the same mass (squared) M^2. The $\mathcal{N} = 1$ representations of the supersymmetric algebra for massive states is given in Table

Spin	Ω_0	$\Omega_{\frac{1}{2}}$
0	2	1
$\frac{1}{2}$	1	2
1		1

Table 2.1: Massive $\mathcal{N} = 1$ representations

λ	Ω_0	$\Omega_{\frac{1}{2}}$
0	1	
$\frac{1}{2}$	1	1
1		1

Table 2.2: Massless $\mathcal{N} = 1$ representations

2.1, and massless states is given in Table 2.2 [27]. Ω denotes the Clifford vacuum state, with the subscript standing for its spin in the massive case and helicity in the massless state. For a theory to be CPT invariant the number of massless states is doubled as CPT changes the sign of helicity [29].

2.2 Supersymmetric Field Theory

To construct a supersymmetric field theory, the supersymmetry representations must be presented in terms of field operators (fields) which are not restricted by any mass-shell conditions. By defining the anticommuting Grassmann parameters ξ^α and $\bar{\xi}_{\dot{\alpha}}$,

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = \cdots = [\xi^\alpha, P_\mu] = 0 \tag{2.2.1}$$

the entire supersymmetry algebra is defined in terms of bosonic quantities.[1]

$$[\xi Q, \bar{\eta}\bar{Q}] = 2\xi\sigma^\mu\bar{\eta}P_\mu \tag{2.2.2a}$$

$$[P^\mu, \xi Q] = 0 = [P^\mu, \bar{\xi}\bar{Q}] \tag{2.2.2b}$$

$$[M^{\mu\nu}, \xi Q] = -(\xi\sigma^{\mu\nu}Q) \tag{2.2.2c}$$

$$[M^{\mu\nu}, \bar{\xi}\bar{Q}] = -(\bar{\xi}\bar{\sigma}^{\mu\nu}\bar{Q}) \tag{2.2.2d}$$

[1] $\xi\eta = \xi^\alpha\eta_\alpha, \bar{\xi}\bar{\eta} = \bar{\xi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}$.

A finite supersymmetry transformation is generated by the unitary operator [42],

$$U = e^{i(\xi Q+\bar{\xi}\bar{Q})} \tag{2.2.3}$$

acting on a field. The corresponding infinitesimal transformation on a field is,

$$\delta_\xi \phi(x) = \left[i(\xi Q + \bar{\xi}\bar{Q}), \phi(x)\right] \tag{2.2.4}$$

The supersymmetry transformation transforms tensor fields into spinor fields and vice versa. To present the irreducible supersymmetry representations in terms of fields, consider a scalar field $\varphi(x)$. In a four-dimensional spacetime the complex scalar field has mass dimension $[\varphi] = 1$, from the supersymmetry algebra the supercharges have mass dimension $[Q] = [\bar{Q}] = \frac{1}{2}$ and by equation (2.2.2) the Grassmann parameters have mass dimension $[\xi] = [\bar{\xi}] = -\frac{1}{2}$. Based on the mass dimensions of the scalar field, supercharges and Grassmann parameters, the simplest infinitesimal supersymmetry transformation of the scalar field is [28],

$$\delta_\xi \varphi = a\xi\psi + b\bar{\xi}\bar{\psi} \tag{2.2.5}$$

The spinor fields $\psi(x)$, $\bar{\psi}(x)$ have mass dimension $[\psi] = [\bar{\psi}] = \frac{3}{2}$. The spinor field $\psi(x)$ must also have a supersymmetry transformation and in order to obtain a supersymmetric field theory, the algebra must close. A supersymmetry transformation of the spinor $\delta_\xi\psi$ will have mass dimension $\frac{3}{2}$. To close the algebra with only the two fields $\varphi(x)$ and $\psi(x)$ the supersymmetry transformation of the spinor $\psi(x)$ must be proportional to a derivative of the scalar field.

$$\delta_\xi \psi_\alpha = c(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_\mu\varphi \qquad \delta_\xi \bar{\psi}^{\dot{\alpha}} = c^*(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\xi_\alpha\partial_\mu\varphi^* \tag{2.2.6}$$

The unknown coefficients a, b and c in equations (2.2.5) and (2.2.6) are determined by the superalgebra. Using the definition of a supersymmetry transformation on a scalar field $\varphi(x)$ (equation (2.2.4)), the superalgebra (2.1.7) and the Jacobi identity (2.1.1),

$$\begin{aligned}(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\varphi(x) &= -\left(2(\eta\sigma^\mu\bar{\xi}) - 2(\xi\sigma^\mu\bar{\eta})\right)[P_\mu, \varphi(x)] \\ &= i\left(2(\eta\sigma^\mu\bar{\xi}) - 2(\xi\sigma^\mu\bar{\eta})\right)\partial_\mu\varphi(x)\end{aligned} \tag{2.2.7}$$

Performing the same calculation using the transformations (2.2.5) and (2.2.6),

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\varphi(x) = ac(\xi\sigma^\mu\bar{\eta} - \eta\sigma^\mu\bar{\xi})\partial_\mu\varphi + bc^*(\bar{\xi}\bar{\sigma}^\mu\eta - \bar{\eta}\bar{\sigma}^\mu\xi)\partial_\mu\varphi \tag{2.2.8}$$

For the transformations (2.2.5) and (2.2.6) to be valid supersymmetry transformations, equations (2.2.7) and (2.2.8) must be equal. This equivalence is found for $ac = 2i$ and $b = 0$, unless $\varphi(x)$ is a constant field. This same procedure can be performed for the spinor field. From the definition of a supersymmetry transformation (equation (2.2.4)), the superalgebra (2.1.7) and the Jacobi identity (2.1.1),

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\,\psi_\alpha = 2i\left(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta}\right)\partial_\mu\psi_\alpha(x) \tag{2.2.9}$$

This same calculation using the transformations (2.2.5) and (2.2.6),

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\,\psi_\alpha = -ac\,\left(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta}\right)\left(\partial_\mu\psi_\alpha - \frac{1}{2}\left(\sigma_\mu\bar{\sigma}^\nu\partial_\nu\psi\right)_\alpha\right) \tag{2.2.10}$$

This is only equivalent to equation (2.2.9) if $\bar{\sigma}^\mu\partial_\mu\psi_\alpha = 0$, the algebra can only be closed if this condition is met.

Consider a free theory of one complex scalar field $\varphi(x)$ and one spinor field $\psi(x)$. The Lagrangian is,

$$\mathcal{L} = -\partial_\mu\varphi^\dagger\partial^\mu\varphi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \tag{2.2.11}$$

Under a supersymmetry transformation the Lagrangian transforms as,

$$\begin{aligned}\delta_\xi\mathcal{L} = &-(a - ic^\dagger)(\partial_\mu\varphi^\dagger)\xi\partial^\mu\psi + (ic + a^\dagger)(\bar{\xi}\partial_\mu\bar{\psi})\partial^\mu\varphi \\ &- i\partial_\mu\left\{c(\bar{\xi}\bar{\psi})(\partial^\mu\varphi) - 2c^\dagger(\xi\sigma^{\nu\mu}\psi)\partial_\nu\varphi^\dagger\right\}\end{aligned} \tag{2.2.12}$$

A total divergence does not contribute to the action of a field theory, therefore it is invariant under supersymmetry transformations if $a = ic^\dagger$. The equations of motion are obtained via the functional Euler-Lagrange equation.

$$\frac{\mathcal{D}\mathcal{L}}{\mathcal{D}\phi(x)} - \partial_\mu\frac{\mathcal{D}\mathcal{L}}{\mathcal{D}(\partial_\mu\phi(x))} = 0 \tag{2.2.13}$$

which gives,

$$\partial_\mu\varphi^\dagger(x) = 0 \tag{2.2.14}$$

$$\bar{\sigma}^\mu\partial_\mu\psi = 0 \tag{2.2.15}$$

The condition to close the field algebra can be satisfied by the equation of motion for a free spinor. This is *on-shell* supersymmetry. The equations of motion must be used to close the algebra. This is due to the bosonic and fermionic degrees of freedom being unequal. A complex scalar has two real degrees of freedom and a Weyl spinor ψ has four real degrees of freedom (two from each $\alpha = 1, 2$). By imposing the condition (2.2.15) and going on-shell, the number of fermionic degrees of freedom is restricted to 2.

In order to have a supersymmetric quantum field theory, supersymmetry must be realised *off-shell*. Off-shell supersymmetry is achieved by introducing an additional scalar field. The bosonic and fermionic degrees of freedom are now equal. The additional field must remove the unwanted contribution to equation (2.2.10) and not contribute to equation (2.2.8). With the additional scalar field the supersymmetry transformation (2.2.6) can have an additional term,

$$\delta_\xi \psi = c\sigma^\mu \bar{\xi} \partial_\mu \varphi + e\xi F \tag{2.2.16}$$

This transformation does not contribute to equation (2.2.8) for constant e, so the algebra is closed with respect to $\varphi(x)$. Consider the transformation,

$$\delta_\eta \delta_\xi \psi_\alpha = \delta_\eta \left(c\sigma^\mu \bar{\xi} \partial_\mu \varphi + e\xi F \right) \tag{2.2.17}$$

The scalar field $F(x)$ must also have a supersymmetry transformation. $F(x)$ has dimension $[F] = 2$. As with the transformation of the spinor field $\psi(x)$, to close the algebra $\delta_\xi F$ must be proportional to the derivative of the spinor, $\partial_\mu \psi$.

$$\delta_\xi F = f\bar{\xi}\bar{\sigma}^\mu \partial_\mu \psi \tag{2.2.18}$$

With this transformation, equation (2.2.10) becomes,

$$\begin{aligned} (\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\, \psi_\alpha = -ac\, (\eta\sigma^\mu \bar{\xi} - \xi\sigma^\mu \bar{\eta}) \big(\partial_\mu \psi_\psi - \frac{1}{2}\, (\sigma_\mu \bar{\sigma}^\nu \partial_\nu \psi)_\alpha \big) \\ - \frac{1}{2}\, ef\, (\eta\sigma_\nu \bar{\xi} - \xi\sigma_\nu \bar{\eta})(\sigma^\nu \bar{\sigma}^\mu \partial_\mu \psi)_\alpha \end{aligned} \tag{2.2.19}$$

This is equivalent to equation (2.2.7) for $ac = ef$, so the algebra can be closed. With the unknown constants determined the supersymmetry transformations of the fields

$\varphi(x)$, $\psi(x)$ and $F(x)$ are,

$$\delta_\xi \varphi = \sqrt{2}\,\xi\psi \tag{2.2.20a}$$
$$\delta_\xi \psi = i\sqrt{2}\,\sigma^\mu\bar{\xi}\partial_\mu\varphi + \sqrt{2}\,\xi F \tag{2.2.20b}$$
$$\delta_\xi F = i\sqrt{2}\,\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi \tag{2.2.20c}$$

The individual fields $\varphi(x), \psi(x)$ and $F(x)$ are called component fields and they form a 'component' multiplet. This particular multiplet is the *chiral* multiplet, the massless representation obtained from Ω_0 in Table 2.2. The Lagrangian for this free massless multiplet is,

$$\mathcal{L} = -\partial_\mu\varphi^\dagger\partial^\mu\varphi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F \tag{2.2.21}$$

The scalar field F has no kinetic term, so it does not propagate. It is an *auxilary field*, which can be eliminated from the Lagrangian by its equation of motion. On-shell, the massless chiral multiplet contains a spin-0 particle and a spin-$\frac{1}{2}$ particle, whilst a massive chiral multiplet contains two spin-0 particles and a spin-$\frac{1}{2}$ particle.

2.3 Superfields

Supersymmetric field theories are the same as their non-supersymmetric cousins, except they possess an additional symmetry, they are invariant under supersymmetry transformations. The superfield formulation of supersymmetric field theories provides a compact description of $\mathcal{N} = 1$ supersymmetry representations and is an effective tool for constructing supersymmetric Lagrangians. A supermultiplet of bosonic and fermionic fields can be represented as a single field on a superspace with coordinates $z = z(x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$, where θ and $\bar{\theta}$ are Grassmann coordinates. A group element of the supersymmetry algebra can be defined using the mathematics of coset spaces [42].

$$G(x,\theta,\bar{\theta}) = e^{i(x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q})} \tag{2.3.1}$$

Two group elements can be multiplied together using the Baker-Campbell-Hausdorff formula [27],

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots} \tag{2.3.2}$$

A supersymmetry transformation $G(0, \xi, \bar{\xi})$ of a group element is,

$$G(0,\xi,\bar{\xi})G(x^\mu,\theta,\bar{\theta}) = G(x^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}, \xi+\theta, \bar{\xi}+\bar{\theta}) \tag{2.3.3}$$

The supersymmetry transformation induces a motion in the superspace,

$$g(\xi,\bar{\xi}) : (x^\mu,\theta,\bar{\theta}) \to (x^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}, \xi+\theta, \bar{\xi}+\bar{\theta}) \tag{2.3.4}$$

The same motion of the superspace can be generated by a set of differential operators.

$$\begin{aligned} G(x^\mu + & i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}, \xi+\theta, \bar{\xi}+\bar{\theta}) \\ &= G(x^\mu,\theta,\bar{\theta}) + (-i\theta\sigma^\mu\bar{\xi} + i\xi\sigma^\mu\bar{\theta})\frac{\partial G}{\partial x^\mu} + \xi^\alpha\frac{\partial G}{\partial\theta^\alpha} + \bar{\xi}_{\dot{\alpha}}\frac{\partial G}{\partial\bar{\theta}_{\dot{\alpha}}} + \ldots \\ &= e^{\xi Q + \bar{\xi}\bar{Q}}G(x,\theta,\bar{\theta}) \end{aligned} \tag{2.3.5}$$

where the differential operators are,

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{2.3.6a}$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu \tag{2.3.6b}$$

This is a linear representation of the supersymmetry algebra.

A superfield $S(x^\mu,\theta,\bar{\theta})$ is a function of the superspace $z(x^\mu,\theta,\bar{\theta})$ and can be expanded in powers of θ and $\bar{\theta}$ via a Taylor series. This expansion will truncate due to the parameters θ and $\bar{\theta}$ being Grassmann-valued. The coefficents of the expansion are the component fields [27].

$$\begin{aligned} S(x^\mu,\theta,\bar{\theta}) =& F(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) \\ &+ \theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}D(x) \end{aligned} \tag{2.3.7}$$

Superfields are generally reducible and irreducible representations are obtained by imposing constraints on the superfield. The supersymmetry transformations of the component fields can be determined by applying the supersymmetry transformation to the superfield and comparing the terms order by order.

$$\begin{aligned} \delta_\xi S(x^\mu,\theta,\bar{\theta}) =& \delta_\xi F(x) + \theta\,\delta_\xi\phi(x) + \bar{\theta}\,\delta_\xi\bar{\chi}(x) + \theta\theta\,\delta_\xi M(x) + \bar{\theta}\bar{\theta}\,\delta_\xi N(x) \\ &+ \theta\sigma^\mu\bar{\theta}\,\delta_\xi V_\mu(x) + \theta\theta\bar{\theta}\,\delta_\xi\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\,\delta_\xi\psi(x) + \theta\theta\bar{\theta}\bar{\theta}\,\delta_\xi D(x) \\ =& (\xi Q + \bar{\xi}\bar{Q})\,S(x^\mu,\theta,\bar{\theta}) \end{aligned} \tag{2.3.8}$$

Superspace derivatives can be defined which anticommute with the generators of the superalgebra. These are superspace covariant derivatives [27].

$$\begin{aligned} D_\alpha &= \partial_\alpha + i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu && \text{(2.3.9a)} \\ \bar D_{\dot\alpha} &= -\partial_{\dot\alpha} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu && \text{(2.3.9b)} \end{aligned}$$

where $\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}$ and $\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$. They have the algebra,

$$\begin{aligned} \{D_\alpha, \bar D_{\dot\alpha}\} &= 2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu = 2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu && \text{(2.3.10a)} \\ \{D_\alpha, D_\beta\} &= \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0 && \text{(2.3.10b)} \end{aligned}$$

Quantum field theories are non-renormalisable if they contain particles of spin-$\frac{3}{2}$ or greater. There are only two $\mathcal{N} = 1$ multiplets, the chiral and *vector* multiplets, which contain only spin-0, $\frac{1}{2}$ and 1 fields. A chiral superfield Φ is constructed from the constraint $\bar D_{\dot\alpha}\Phi = 0$ [27].

$$\begin{aligned} \Phi = \Phi(x) &+ i\theta\sigma^\mu\bar\theta\,\partial_\mu\Phi(x) + \frac{1}{4}\,\theta\theta\bar\theta\bar\theta\,\partial_\mu\partial^\mu\Phi(x) \\ &+ \sqrt{2}\,\theta\psi(x) - \frac{i}{\sqrt{2}}\,\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar\theta + \theta\theta F(x) \end{aligned} \tag{2.3.11}$$

This multiplet was described by equations (2.2.20) and (2.2.21) as the massless representation Ω_0 of Table 2.2. A vector superfield is constructed from the reality constraint $V = V^\dagger$ [27].

$$\begin{aligned} V(x^\mu, \theta, \bar\theta) = C(x) &+ i\theta\chi(x) - i\bar\theta\bar\chi(x) + \frac{i}{2}\,\theta\theta\,(M(x) + iN(x)) \\ &- \frac{i}{2}\,\bar\theta\bar\theta\,(M(x) - iN(x)) - \theta\sigma^\mu\bar\sigma A_\mu(x) + i\theta\theta\bar\theta\left(\bar\lambda(x) + \frac{i}{2}\,\bar\sigma^\mu\partial_\mu\chi(x)\right) \\ &- i\bar\theta\bar\theta\theta\left(\lambda(x) + \frac{i}{2}\,\sigma^\mu\partial_\mu\bar\chi(x)\right) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(D(x) + \frac{1}{2}\,\partial_\mu\partial^\mu C(x)\right) \end{aligned} \tag{2.3.12}$$

The reality constraint is also valid for the superfield,

$$S(x^\mu, \theta, \bar\theta) = \Phi(x^\mu, \theta, \bar\theta) + \Phi^\dagger(x^\mu, \theta, \bar\theta) \tag{2.3.13}$$

The sum of a chiral and anti-chiral superfield is a vector superfield.

$$\begin{aligned}\Phi + \Phi^\dagger = &\ \Phi(x) + \Phi^\dagger(x) + \sqrt{2}\left(\theta\psi(x) + \bar{\theta}\bar{\psi}(x)\right) + \theta\theta F(x) + \bar{\theta}\bar{\theta}F^\dagger(x) \\ &+ i\theta\sigma^\mu\bar{\theta}\partial_\mu\left(\Phi(x) - \Phi^\dagger(x)\right) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\sigma^\mu\partial_\mu\psi(x) \\ &+ \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\left(\Phi(x) + \Phi^\dagger(x)\right)\end{aligned} \tag{2.3.14}$$

Consider the following transformation of the vector superfield V.

$$V \to V + \Phi + \Phi^\dagger \tag{2.3.15}$$

Under this transformation, the component vector field A_μ transforms as,

$$A_\mu = A_\mu - \partial_\mu\Lambda \tag{2.3.16a}$$

$$\Lambda = i\left(\Phi - \Phi^\dagger\right) \tag{2.3.16b}$$

This is an infinitesimal $U(1)$ gauge transformation of A_μ. This identifies the transformation (2.3.15) as the gauge transformation of a vector superfield. The identification allows the first five component fields of the vector superfield to be set to zero, this is the Wess-Zumino gauge.

$$V_{WZ} = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{2.3.17}$$

where,

$$V(x^\mu, \theta, \bar{\theta}) = V_{WZ} + \Phi + \Phi^\dagger \tag{2.3.18}$$

In the Wess-Zumino gauge the vector superfield has only four component fields. A residual $U(1)$ gauge symmetry remains for the component vector field A_μ, given by (2.3.16). The vector multiplet is the massless representation of supersymmetry obtained from the Clifford vacuum $\Omega_{\frac{1}{2}}$. On-shell, the massless vector multiplet contains a spin-$\frac{1}{2}$ particle and a spin-1 particle, whilst a massive vector multiplet contains one spin-0 particle, two spin-$\frac{1}{2}$ particles and one spin-1 particle.

The abelian gauge transformation can be generalized to a non-abelian gauge transformation. The finite version of the transformation (2.3.15) is [29],

$$e^V \to e^{-i\Lambda^\dagger}e^V e^{i\Lambda} \tag{2.3.19}$$

where $\Phi \longrightarrow i\Lambda$ is a chiral superfield and $\Phi^\dagger \longrightarrow -i\Lambda^\dagger$ is an anti-chiral superfield. A non-abelian gauge transformation is constructed by replacing,

$$V \longrightarrow V^a \tau^a \tag{2.3.20a}$$

$$\Lambda \longrightarrow \Lambda^a \tau^a \tag{2.3.20b}$$

where τ^a are generators of the non-abelian gauge group. To construct a supersymmetric Lagrangian containing vector superfields, a supersymmetric gauge invariant term must be constructed. Consider the chiral superfield,

$$W_\alpha = -\frac{1}{8}\,\bar{D}\bar{D}\,e^{-2V} D_\alpha e^{2V} \tag{2.3.21}$$

Under a gauge transformation the chiral superfield W_α transforms as,

$$W'_\alpha \longrightarrow e^{-2i\Lambda} W_\alpha e^{2i\Lambda} \tag{2.3.22}$$

Therefore the product $W^\alpha W_\alpha$ is invariant under supersymmetric and gauge transformations and can be used to construct a supersymmetric Lagrangian for vector superfields.

2.4 $\mathcal{N} = 1$ Supersymmetric Lagrangians

Superfields are an effective tool for constructing supersymmetric Lagrangians. All renormalisable $\mathcal{N} = 1$ supersymmetric Lagrangians can be described in terms of chiral and vector superfields. Written in terms of superfields, a general $\mathcal{N} = 1$ supersymmetric Lagrangian has the form,

$$\begin{aligned} \mathcal{L} = \frac{1}{8\pi}\Im\left\{\tau_{ym}\int d^2\theta\, W^\alpha W_\alpha\right\} + \int d^2\theta d^2\bar{\theta}\, K(\Phi, \Phi^\dagger, V) \\ + \int d^2\theta\, \mathcal{W}(\Phi) + \int d^2\bar{\theta}\, \bar{\mathcal{W}}(\Phi^\dagger) \end{aligned} \tag{2.4.1}$$

The first term of this Lagrangian is the gauge kinetic term. It describes a pure, non-abelian $\mathcal{N} = 1$ SUSY Yang-Mills theory. The complex gauge coupling is,

$$\tau_{ym} = \frac{\Theta_{ym}}{2\pi} + \frac{4\pi i}{g_{ym}^2} \tag{2.4.2}$$

with the Yang-Mills coupling constant g_{ym} and theta parameter θ_{ym}. The non-abelian supersymmetric field strength superfield W_α is,

$$W_\alpha = -\frac{1}{8}\bar{D}\bar{D}e^{-2V}D_\alpha e^{2V} \tag{2.4.3}$$

In terms of the component fields, the chiral superfield W_α is,

$$\begin{aligned} W_\alpha &= i\lambda_\alpha - \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} - \theta\theta(\sigma^\mu)_{\alpha\dot{\alpha}}D_\mu\bar{\lambda}^{\dot{\alpha}} && \text{(2.4.4a)} \\ W^\alpha &= i\lambda^\alpha - \theta^\alpha D - i(\theta\sigma^{\mu\nu})^\alpha F_{\mu\nu} - \theta\theta\varepsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\alpha}}D_\mu\bar{\lambda}^{\dot{\alpha}} && \text{(2.4.4b)} \end{aligned}$$

The gauge covariant derivative is $D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} + i[A_\mu, \bar{\lambda}]$ and the non-abelian field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. The basis of the gauge kinetic term is the gauge invariant object,

$$\int d^2\theta\, W^\alpha W_\alpha = \mathrm{Tr}\left(-2i\lambda\sigma^\mu D_\mu\bar{\lambda} + D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \tag{2.4.5}$$

where the dual field strength $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$. Inserting the complex gauge coupling (2.4.2), the gauge kinetic term is,

$$\begin{aligned} \frac{1}{8\pi}\Im\left\{\tau_{ym}\int d^2\theta\, W^\alpha W_\alpha\right\} = \mathrm{Tr}\Big\{&-\frac{1}{g_{ym}^2}\, i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2g_{ym}^2}\, D^2 \\ &- \frac{1}{4g_{ym}^2}\, F_{\mu\nu}F^{\mu\nu} - \frac{\Theta_{ym}}{32\pi^2}\, F_{\mu\nu}\tilde{F}^{\mu\nu}\Big\} \end{aligned} \tag{2.4.6}$$

A Lagrangian which contains only the gauge kinetic term describes a pure $\mathcal{N} = 1$ SUSY Yang-Mills theory, a theory of gluons (gauge bosons) and gluinos (gauginos). Matter particles, in either the fundamental or adjoint representations, can be added to a theory through the other terms of the Lagrangian (2.4.1). The gauge-invariant kinetic terms of the chiral multiplets originate from the Kähler potential, the second term in the Lagrangian (2.4.1). The Kähler potential is a real function of chiral, anti-chiral and vector superfields. The Kähler potential provides the most general supersymmetric gauge-invariant kinetic terms for the chiral superfields, but not all potentials lead to a renormalisable theory. Any field theory that contains dimensionful couplings with negative mass dimension is non-renormalisable. The most general

Kähler potential that leads to a renormalisable $\mathcal{N}=1$ supersymmetric Lagrangian is,

$$\begin{aligned}\int d^2\theta d^2\bar{\theta}\ \Phi_i^\dagger e^{2V}\Phi_i = \mathrm{Tr}\Big\{ & -i\psi_i\sigma^\mu D_\mu\bar{\psi}_i - D_\mu\Phi_i^\dagger D^\mu\Phi_i + \frac{1}{\sqrt{2}}i\psi_i[\Phi_i^\dagger,\lambda] \\ & -\frac{1}{\sqrt{2}}i\lambda[\Phi_i^\dagger,\psi_i] - \frac{1}{\sqrt{2}}i\bar{\lambda}[\Phi_i,\bar{\psi}_i] + \frac{1}{\sqrt{2}}i\bar{\psi}_i[\Phi_i,\bar{\lambda}] \\ & - D[\Phi_i^\dagger,\Phi_i] + F_i^\dagger F_i\Big\}\end{aligned} \tag{2.4.7}$$

with the chiral multiplets transforming under the adjoint representation of the gauge group. If the Kähler potential contained any additional chiral superfields then the coupling for the Kähler potential would have a negative mass dimension and hence produce an non-renormalisable theory. The third and fourth terms of the Lagrangian (2.4.1) are the superpotential and its hermitian conjugate. The superpotential describes interactions between the component fields of the chiral multiplets. Different choices of the superpotential lead to different interacting theories. The most general form of a superpotential is,

$$\mathcal{W}(\Phi) = \lambda_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k \tag{2.4.8}$$

where λ, m and g are symmetric coefficients. The coefficient g_{ijk} is a coupling constant, whilst the matrix m_{ij} provides the masses of the fields in the theory. Higher-order terms in the superfield Φ_i do not lead to renormalisable field theories as the complex coefficients have negative mass dimensions.

Inserting the explicit expressions of the gauge kinetic term and the Kähler potential, the most general renormalisable supersymmetric Lagrangian is,

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big\{ & -\frac{1}{4g_{ym}^2}F_{\mu\nu}F^{\mu\nu} - \frac{\Theta_{ym}}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{g_{ym}^2}i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2g_{ym}^2}D^2 \\ & -i\psi_i\sigma^\mu D_\mu\bar{\psi}_i - D_\mu\Phi_i^\dagger D^\mu\Phi_i + \frac{1}{\sqrt{2}}i\psi_i[\Phi_i^\dagger,\lambda] - \frac{1}{\sqrt{2}}i\lambda[\Phi_i^\dagger,\psi_i] \\ & -\frac{1}{\sqrt{2}}i\bar{\lambda}[\Phi_i,\bar{\psi}_i] + \frac{1}{\sqrt{2}}i\bar{\psi}_i[\Phi_i,\bar{\lambda}] - D[\Phi_i^\dagger,\Phi_i] + F_i^\dagger F_i\Big\} \\ & + \int d^2\theta\,\mathcal{W}(\Phi_i) + \int d^2\bar{\theta}\,\bar{\mathcal{W}}(\Phi_i^\dagger)\end{aligned} \tag{2.4.9}$$

The remaining freedom in this Lagrangian is the unspecified superpotential, as different choices of the superpotential lead to different interacting supersymmetric theories. The Lagrangian contains the auxiliary fields F and D to retain off-shell supersymmetry. These can be eliminate by solving the equations of motion for F and D, respectively.

$$F_i = -\frac{\partial\bar{\mathcal{W}}(\Phi_i^\dagger)}{\partial\Phi_i^\dagger} \tag{2.4.10a}$$

$$F_i^\dagger = -\frac{\partial\mathcal{W}(\Phi_i)}{\partial\Phi_i} \tag{2.4.10b}$$

$$D = g_{ym}^2[\Phi_i^\dagger, \Phi_i] \tag{2.4.10c}$$

The D field can be eliminated without reference to the superpotential, therefore,

$$\mathrm{Tr}\left\{\frac{1}{2g_{ym}^2}D^2 - D[\Phi_i^\dagger, \Phi_i]\right\} = -\frac{1}{2}g_{ym}^2\mathrm{Tr}\,[\Phi_i^\dagger, \Phi_i]^2 \tag{2.4.11}$$

which can be substituted in the Lagrangian (2.4.9). The F field can only be eliminated once the superpotential has been specified. In Section 2.6 the full Lagrangian of the $\mathcal{N} = 1^*$ SUSY Yang-Mills theory will be constructed after defining its superpotential.

Supersymmetric Vacua

Supersymmetry is spontaneously broken if the vacuum is not invariant under a supersymmetry transformation [27].

$$Q_\alpha|0> \neq 0 \tag{2.4.12}$$

Supersymmetry will be spontaneously broken if a field has a non-zero vacuum expectation value. Only scalar fields may acquire a vacuum expectation value. Fields which transform as a spinor or vector under a Lorentz transformation violate Lorentz invariance if they acquire a vacuum expectation value. Consider a chiral multiplet with the supersymmetry transformations [27],

$$\delta_\xi\Phi = \sqrt{2}\,\xi\psi \tag{2.4.13a}$$

$$\delta_\xi\psi = i\sqrt{2}\,\sigma^\mu\bar{\xi}\partial_\mu\Phi + \sqrt{2}\,\xi F \tag{2.4.13b}$$

$$\delta_\xi F = i\sqrt{2}\,\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi \tag{2.4.13c}$$

The only non-zero vacuum expectation value that will break supersymmetry is given by,

$$< 0|\delta_\xi \psi|0 >= \sqrt{2}\,\xi < 0|F|0 > \tag{2.4.14}$$

Note that the other term in this expression, involving $< 0|\partial_\mu \Phi|0 >$, must be zero in order to preserve Lorentz invariance. Classically, a supersymmetric theory with a chiral multiplet will preserve supersymmetry if,

$$F = -\frac{\partial \bar{\mathcal{W}}}{\partial \Phi^\dagger} = 0 \qquad\qquad F^\dagger = -\frac{\partial \mathcal{W}}{\partial \Phi} = 0 \tag{2.4.15}$$

This is the F-flatness condition. Consider a vector multiplet with the supersymmetry transformations [27],

$$\delta_\xi A^\mu = -i\bar{\lambda}\bar{\sigma}^\mu \xi + i\bar{\xi}\sigma^\mu \lambda \tag{2.4.16a}$$
$$\delta_\xi \lambda = \sigma^{\mu\nu}\xi F_{\mu\nu} + i\xi D \tag{2.4.16b}$$
$$\delta_\xi D = -\xi\sigma^\mu D_\mu \bar{\lambda} - D_\mu \lambda \sigma^\mu \bar{\xi} \tag{2.4.16c}$$

The only non-zero vacuum expectation value that will break supersymmetry is given by,

$$< 0|\delta_\xi \lambda|0 >= i\xi < 0|D|0 > \tag{2.4.17}$$

Classically, a supersymmetric theory with a vector multiplet will preserve supersymmetry if,

$$D = -g_{ym}^2 [\Phi\ , \Phi^\dagger] = 0 \tag{2.4.18}$$

This is the D-flatness condition.

The scalar potential of a supersymmetric theory is,

$$\mathcal{V} = F^\dagger F\ + \frac{1}{2} D^2 \tag{2.4.19}$$

Supersymmetry is preserved in a field theory if the F-flatness and D-flatness conditions are satisfied. Consequently, the scalar potential must also be zero. Supersymmetry is broken if a theory possesses a non-supersymmetric minima of the scalar potential [28]. A gauge symmetry can also be spontaneously broken by a scalar field acquiring a non-zero vacuum expectation value. Within the chiral multiplet, supersymmetry is not broken if the scalar field Φ has a non-zero expectation value.

$$< 0|\Phi|0 > \neq 0 \tag{2.4.20}$$

However this expectation value is sufficient to break the gauge group of a supersymmetric theory, via the Higgs mechanism. It is possible for supersymmetry and gauge symmetry to be broken independently or simultaneously.

2.5 Extended Supersymmetry and Supergravity

This Chapter has focused on $\mathcal{N} = 1$ supersymmetry representations and field theories in four spacetime dimensions. Supersymmetry is not limited to these applications. $\mathcal{N} = 1$ supersymmetry is the minimal amount of supersymmetry in four dimensions and contains four supercharges $Q_{(1,2)}$, $\bar{Q}_{(\dot{1},\dot{2})}$. If supersymmetry is present in nature, then it must be broken at a scale beyond the standard model. In this case theories with minimial supersymmetry are of the most phenomenological interest. The superalgebra presented in Section 2.1 allows theories with a larger number of supercharges to be considered. A theory with minimal supersymmetry has a $U(1)$ R-symmetry, whilst a theory with extended supersymmetry has a larger R-symmetry. The more (super)symmetries present in a physical theory, the greater the constraints on the theory. The study of theories with extended supersymmetry has revealed highly constrained theories which are exactly solvable, such as $\mathcal{N} = 4$ SUSY Yang-Mills theory [3]. The maximal amount of supersymmetry is dictated by renormalisation, which implies that the largest number of allowed supersymmetries is sixteen, or $\mathcal{N} = 4$ supersymmetry [3]. It is always possible to consider a system with extended supersymmetry in terms of $\mathcal{N} = 1$ supersymmetry multiplets. This Thesis makes reference to two theories with extended supersymmetry, the $\mathcal{N} = 4$ SUSY Yang-Mills theory and the $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory in six spacetime dimensions. Supersymmetry in six dimensions is chiral hence the notation (1,1). Supersymmetry in diverse dimensions will be discussed in the next Chapter.

Supersymmetry can also be considered as a local symmetry rather than a global symmetry. Under local supersymmetry, the Grassmann parameters $\xi, \bar{\xi}$ become functions of spacetime position. A theory invariant under local supersymmetry transformations is also invariant under general coordinate transformations, therefore local supersymmetry is a theory of gravity, called supergravity [28]. In supergravity theories, the maximal amount of supersymmetry is 32 supercharges or $\mathcal{N} = 8$ supersymmetry in four-dimensional spacetime. Supergravity theories are important as they are the low-energy description ($\alpha' \to 0$) of some closed string theories and are essential to our understanding of the AdS/CFT correspondence.

2.6 $\mathcal{N} = 1^*$ SUSY Yang-Mills Theory

This Chapter has presented the mathematical framework of supersymmetry and, in particular, has shown how to construct the Lagrangian of a field theory possessing $\mathcal{N} = 1$ supersymmetry, using $\mathcal{N} = 1$ superfields. In the final Section of this Chapter, the framework presented will be used to construct the Lagrangian of a specific theory, the $\mathcal{N} = 1^*$ SUSY Yang-Mills theory. The $\mathcal{N} = 1^*$ theory is defined as a relevant deformation of the $\mathcal{N} = 4$ SUSY Yang-Mills theory. The $\mathcal{N} = 4$ SUSY Yang-Mills theory comprises of a single massless $\mathcal{N} = 4$ vector multiplet. In terms of $\mathcal{N} = 1$ supersymmetry, this comprises of a massless $\mathcal{N} = 1$ vector multiplet and three massless $\mathcal{N} = 1$ chiral multiplets. The $\mathcal{N} = 1^*$ theory is constructed by softly breaking the $\mathcal{N} = 4$ supersymmetry by adding mass terms for the chiral multiplets in the $\mathcal{N} = 4$ superpotential [22]. The $\mathcal{N} = 1^*$ theory comprises of a massless $\mathcal{N} = 1$ vector multiplet and three massive $\mathcal{N} = 1$ chiral multiplets. The superpotential of the $\mathcal{N} = 1^*$ SUSY Yang-Mills theory is,

$$\mathcal{W}(\Phi) = g_{ym} \operatorname{Tr} \left\{ \frac{\sqrt{2}}{6} \, i\varepsilon_{ijk} \Phi_i [\Phi_j, \Phi_k] + \frac{m_i}{g_{ym}} \, \Phi_i^2 \right\} \tag{2.6.1}$$

Each chiral multiplet (labelled by the index $i = 1, 2, 3$) has a different mass m_i, which for simplicity are set to be equal, $m_i = \eta$. As the theory originated from the $\mathcal{N} = 4$ theory and contained a single massless vector multiplet of $\mathcal{N} = 4$ supersymmetry, the chiral multiplets of the $\mathcal{N} = 1^*$ theory are adjoint representations of the gauge group $U(N)$ (or $SU(N)$).

From the superpotential (2.6.1) the remaining terms in the Lagrangian (2.4.9) can be determined. The equations of motion of the auxiliary scalar fields F_i are,

$$F_i = \frac{1}{\sqrt{2}} i g_{ym} \varepsilon_{ijk} [\Phi_k^\dagger, \Phi_j^\dagger] - 2\eta \Phi_i^\dagger \tag{2.6.2a}$$

$$F_i^\dagger = -\frac{1}{\sqrt{2}} i g_{ym} \varepsilon_{ijk} [\Phi_j, \Phi_k] - 2\eta \Phi_i \tag{2.6.2b}$$

The elimination of the auxiliary fields F_i in the Lagrangian has the contribution,

$$\begin{aligned} F_i^\dagger F_i = \mathrm{Tr}\Big\{ & g_{ym}^2 [\Phi_j^\dagger, \Phi_i^\dagger][\Phi_i, \Phi_j] + \sqrt{2} i g_{ym} \eta \varepsilon_{ijk} [\Phi_i^\dagger, \Phi_j^\dagger] \Phi_k \\ & + \sqrt{2} i g_{ym} \eta \varepsilon_{ijk} \Phi_i^\dagger [\Phi_j, \Phi_k] + 4\eta^2 \Phi_i^\dagger \Phi_i \Big\} \end{aligned} \tag{2.6.3}$$

The superpotential contribution to the Lagrangian is found to be,

$$\begin{aligned} & \int d^2\theta\, \mathcal{W}(\Phi) + \int d^2\bar{\theta}\, \bar{\mathcal{W}}(\Phi^\dagger) \\ & = \mathrm{Tr}\Big\{ 2 g_{ym}^2 [\Phi_i^\dagger, \Phi_j^\dagger][\Phi_i, \Phi_j] - 2\sqrt{2}\, i g_{ym} \eta \varepsilon_{ijk} [\Phi_i^\dagger, \Phi_j^\dagger] \Phi_k \\ & \quad - 2\sqrt{2}\, i g_{ym} \eta \varepsilon_{ijk} \Phi_i^\dagger [\Phi_j, \Phi_k] - 8\eta^2 \Phi_i^\dagger \Phi_i + \frac{1}{\sqrt{2}} i g_{ym} \varepsilon_{ijk} \psi_i [\Phi_k, \psi_j] \\ & \quad + \frac{1}{\sqrt{2}} i g_{ym} \varepsilon_{ijk} \bar{\psi}_i [\Phi_k^\dagger, \bar{\psi}_j] - \eta \psi_i \psi_i - \eta \bar{\psi}_i \bar{\psi}_i \Big\} \end{aligned} \tag{2.6.4}$$

Inserting these contributions into the general Lagrangian (2.4.9), the Lagrangian for the $\mathcal{N} = 1^*$ theory is,

$$\begin{aligned} \mathcal{L} = \mathrm{Tr}\Big\{ & -\frac{1}{4 g_{ym}^2} F_{\mu\nu} F^{\mu\nu} - \frac{\Theta_{ym}}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{1}{g_{ym}^2} i \lambda \sigma^\mu D_\mu \bar{\lambda} - i \psi_i \sigma^\mu D_\mu \bar{\psi}_i \\ & - D_\mu \Phi_i^\dagger D^\mu \Phi_i + \frac{1}{\sqrt{2}} i \psi_i [\Phi_i^\dagger, \lambda] - \frac{1}{\sqrt{2}} i \lambda [\Phi_i^\dagger, \psi_i] - \frac{1}{\sqrt{2}} i \bar{\lambda} [\Phi_i, \bar{\psi}_i] \\ & + \frac{1}{\sqrt{2}} i \bar{\psi}_i [\Phi_i, \bar{\lambda}] - \frac{1}{2} g_{ym}^2 [\Phi_i, \Phi_i^\dagger]^2 - g_{ym}^2 [\Phi_j^\dagger, \Phi_i^\dagger][\Phi_i, \Phi_j] \\ & - \sqrt{2}\, i g_{ym} \eta \varepsilon_{ijk} [\Phi_i^\dagger, \Phi_j^\dagger] \Phi_k - \sqrt{2}\, i g_{ym} \eta \varepsilon_{ijk} \Phi_i^\dagger [\Phi_j, \Phi_k] - 4\eta^2 \Phi_i^\dagger \Phi_i \\ & + \frac{1}{\sqrt{2}} i g_{ym} \varepsilon_{ijk} \psi_i [\Phi_k, \psi_j] + \frac{1}{\sqrt{2}} i g_{ym} \varepsilon_{ijk} \bar{\psi}_i [\Phi_k^\dagger, \bar{\psi}_j] - \eta \psi_i \psi_i - \eta \bar{\psi}_i \bar{\psi}_i \Big\} \end{aligned} \tag{2.6.5}$$

The $\mathcal{N} = 1^*$ theory has an enlarged R-symmetry of $SO(6) \sim SU(4)$ which is inherited from the $\mathcal{N} = 4$ theory. The full R-symmetry is partially hidden by the $\mathcal{N} = 1$ superfield notation, with only the $SU(3) \times U(1)$ subgroup manifest [22].

Chapter 3

Extra Dimensions

Our observed universe has four spacetime dimensions. The dimensionality of our universe can be determined by the examination of both the electromagnetic and gravitational forces. For a universe with D spacetime dimensions ($D-1$ space-like, one time-like), both the electromagnetic and gravitational forces would obey $F \sim R^{-(D-2)}$. Electromagnetism and gravity obey an inverse square law, therefore the universe has four spacetime dimensions [5]. Despite the evidence that the universe has four dimensions, physicists have continued to studied theories with extra dimensions. Some have studied higher-dimensional field theories as toy models, whilst others study them from a phenomenological perspective, such as stablising the electroweak scale of the standard model [32]. String theory is only consistent in a ten-dimensional spacetime, therefore the use of string theory as an effective theory of confining gauge theories requires the study of higher-dimensional theories. To study a higher-dimensional theory within a phenomenological context then the presence of extra dimensions must be resolved with the observation that the universe is four-dimensional. Section 3.1 discusses the construction of higher-dimensional field theories. The generalisation of scalar and vector fields to an arbitrary dimension is trivial, however the treatment of spinors (and consequently supersymmetry) in an arbitary dimension is rather non-trivial. Section 3.2 discusses the realisation of higher-dimensional field theories as phenomenological theories through the dimensional reduction and compactification of one or more dimensions. In compactifying a theory the spacetime manifold becomes curved, so Section 3.3 discusses field theories in curved spacetime. Section 3.4 discusses the spherical compactification of a gauge theory in detail. Finally, Section 3.5 discusses the preservation of supersymmetry in a compactification.

3.1 Higher-Dimensional Field Theories

A field theory can be constructed in any spacetime dimension. This Section will discuss the construction of higher-dimensional field theories and the consequences to supersymmetry.[1] The discussion will focus on scalar, spinor and vector fields in higher-dimensional theories, although the arguments could be extended to discuss fields with spin > 1. The spacetime will be taken to have D dimensions. Any point in this D-dimensional spacetime can be specified through a coordinate vector with D components labelling each dimension x^a, $a = 0, 1, \ldots, D-1$. The additional dimensions are considered to be spatial, identical to the three spatial dimensions already considered. Consequently, the spacetime remains isotropic and enlarges the isotropy group of Lorentz transformations to $SO(D-1,1)$.

Scalar and Vector Fields in Diverse Dimensions

The generalisation of scalar and vector fields to an arbitrary dimension is trivial, the spacetime index runs over all dimensions. A scalar field $\phi(x)$ takes a value at all points in spacetime and it is defined as a field that remains invariant under a Lorentz transformation $x^a \longrightarrow x'^a = \Lambda^a{}_b x^b$ [2].

$$\phi(x) \rightarrow \phi'(x') = \phi'(\Lambda\, x) = \phi(x) \tag{3.1.1}$$

A scalar field is a solution of the Klein-Gordan equation,

$$\left(\partial^a \partial_a + m^2\right) \phi(x) = 0 \tag{3.1.2}$$

which is the equation of motion for the action,

$$\mathcal{S} = \int d^D x \left(-\partial_a \phi(x) \partial^a \phi(x) + m^2 \phi(x)^2\right) \tag{3.1.3}$$

[1]The discussions of spinors are based on the discussion in [3].

The action of a field theory must be dimensionless, therefore the mass dimension of the scalar field can be determined. As the dimension of the integral measure is $[d^D x] = -D$ and $[\partial_a] = 1$ then,

$$[\phi(x)] = \frac{1}{2}(D-2) \qquad \text{and} \qquad [m] = 1 \tag{3.1.4}$$

Under the Lorentz transformation $x^a \to x'^a = \Lambda^a{}_b x^b$, a vector field transforms as,

$$A_a(x) \to A'_a(x') = \Lambda_a{}^b A_b(x) \tag{3.1.5}$$

A gauge field satisfies Maxwell's equation,[2]

$$\partial_a \left(\partial^a A^b(x) - \partial^b A^a(x)\right) = 0 \tag{3.1.6}$$

which is the equation of motion for the action,

$$\mathcal{S} = -\frac{1}{4}\int d^D x\, F_{ab}F^{ab} \tag{3.1.7}$$

where the field tensor $F_{ab} = \partial_a A_b(x) - \partial_b A_a(x) + ig[A_a(x), A_b(x)]$ for a non-abelian gauge theory. From the action, the dimension of the vector field is found to be,

$$[A_M(x)] = \frac{1}{2}(D-2) \tag{3.1.8}$$

Spinor Fields and Supersymmetry in Diverse Dimensions

Supersymmetry was introduced in Chapter 2 within the context of four-dimensional field theories. The supersymmetry generators were 4-component spinors described in terms of two 2-component Weyl spinors [27, 28]. The number of components a spinor possesses (and hence the number of supersymmetries) is determined by the dimensionality of the spacetime.

Spinors are non-tensorial representations of the Lorentz group $SO(D)$ or $SO(D-1,1)$. They are often referred to as the square root of a vector, as the direct product of two spinor representations is a tensor representation of the Lorentz group. For

[2]Vector fields in a renormalisable field theory must be gauge fields, so these discussions focus on gauge fields rather than vector fields in general.

example, the direct product of two spinor (spin-$\frac{1}{2}$) representations of $SO(3) \sim SU(2)$ is the direct sum of a vector and scalar representation.

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$$

The Lie algebra of the Lorentz group $SO(D-1,1)$ with metric η_{ab} is,

$$[M^{ab}, M^{cd}] = (\eta^{bd} M^{ac} - \eta^{bc} M^{ad} - \eta^{ad} M^{bc} + \eta^{ac} M^{bd}) \tag{3.1.9}$$

where the spacetime index $a = 0, 1, \ldots, D-1$. A spinor representation of the Lorentz group is constructed from a *Clifford algebra*, a set of operators satisfying the anti-commutation relation,

$$\{\Gamma^a, \Gamma^b\} = -2\eta^{ab}\mathbb{1} \tag{3.1.10}$$

The Clifford algebra forms a representation of the Lorentz group $SO(D-1,1)$,

$$M^{ab} = \frac{1}{4}[\Gamma^a, \Gamma^b] \tag{3.1.11}$$

The Dirac gamma matrices Γ^a are irreducible representations of the Clifford algebra. The Clifford algebra of the Lorentz group $SO(D)$ is related to the Clifford algebra of $SO(D-1,1)$ by the identification $\Gamma^D = i\Gamma^0$.

$$\begin{aligned} \{\Gamma^a, \Gamma^b\} &= 2\delta^{ab}\mathbb{1} && (3.1.12) \\ M^{ab} &= -\frac{i}{4}[\Gamma^a, \Gamma^b] && (3.1.13) \end{aligned}$$

In even dimensions all the irreducible representations of the Clifford algebra are equivalent. They are represented by $N \times N$ matrices, with $N = 2^{D/2}$. Each irreducible representation is related via a similarity transformation.

$$\Gamma'^a = S\Gamma^a S^{-1} \tag{3.1.14}$$

In odd dimensions there are two equivalence classes of irreducible representations, $\{\Gamma^a\}$ and $\{-\Gamma^a\}$, with $N = 2^{(D-1)/2}$. In even dimensions there is an additional matrix which anticommutes with all gamma matrices Γ^a.

$$\Gamma^{D+1} = \Gamma^0 \ldots \Gamma^{D-1} \tag{3.1.15}$$

In odd dimensions the same matrix commutes with all the other gamma matrices.

$$[\Gamma^{D+1}, \Gamma^a] = 0 \tag{3.1.16}$$

By Schur's lemma Γ^{D+1} must be a multiple of the unit matrix [52]. If two irreducible representations are equivalent they will be related by the similarity transform,

$$S\Gamma^a S^{-1} = -\Gamma^a \tag{3.1.17}$$

Applying this similarity transformation to the gamma matrix Γ^{D+1},

$$S\Gamma^{D+1}S^{-1} = (-1)^D\Gamma^{D+1} \tag{3.1.18}$$

In odd dimensions $\Gamma^{D+1} \propto \mathbb{1}$, so equation (3.1.18) gives a contradiction, therefore the irreducible representations $\{\Gamma_a\}$ and $\{-\Gamma^a\}$ are inequivalent. Alternatively in even dimensions equation (3.1.17) is satisfied by $S = \Gamma^{D+1}$ and so $\{\Gamma^a\}$ and $\{-\Gamma^a\}$ are equivalent representations. The similarity matrices are called intertwiners. The remaining discussion will focus on only even dimensions.

The equivalent representations of the Dirac algebra are,

$$\Gamma^a,\ -\Gamma^a,\ (\Gamma^a)^\dagger,\ -(\Gamma^a)^\dagger,\ (\Gamma^a)^T,\ -(\Gamma^a)^T,\ (\Gamma^a)^*,\ -(\Gamma^a)^* \tag{3.1.19}$$

These irreducible representations are related by different similarity transformations.

$$A\Gamma^a A^{-1} = (\Gamma^a)^\dagger \tag{3.1.20}$$

$$C^{-1}\Gamma^a C = -(\Gamma^a)^T \tag{3.1.21}$$

$$\Gamma^{D+1}\Gamma^a(\Gamma^{D+1})^{-1} = -\Gamma^a \tag{3.1.22}$$

These intertwiners can be combined to obtain the remaining irreducible representations. For example, if $D = CA^T$,

$$D^{-1}\Gamma_a D = -\Gamma_a^* \tag{3.1.23}$$

There are some non-trivial relationships between the intertwiners and their transposes, complex conjugates etc [3].

$$A = \alpha\, A^\dagger,\ \Gamma_{D+1} = \beta\, \Gamma_{D+1}^{-1},\ C = \eta\, C^T,\ D = \delta\, (D^{-1})^* \tag{3.1.24}$$

The dimension of the Clifford algebra representations grow exponentially with respect to the spacetime dimension. For even dimensions, the dimension of the Clifford algebra representation is,

$$N = 2^{D/2}$$

As the dimension of the spinor increases exponentially with spacetime dimensions D, it useful to find the smallest (minimal) representation for a given spacetime dimension. Two conditions can be applied to reduce the dimensionality of a spinor representation, the chirality condition and the reality (Majorana) condition. The gamma matrix Γ^{D+1} commutes with M_{ab} and allows the definition of the chirality projections,

$$\frac{1}{2} M^{\pm}_{ab} = \frac{1}{2}(\mathbb{1} \pm \sqrt{\beta}\,\Gamma^{D+1})\frac{1}{2} M_{ab} \tag{3.1.25}$$

The projections are representations of the Clifford algebra and hence the Lorentz group. Applying the chirality condition to a spinor of $SO(D-1,1)$ reduces the dimension of the representation by half. The chirality condition can only be applied in even dimensions [3]. The Majorana condition for a spinor Ψ is,

$$\Psi = \Psi^c = C\bar{\Psi}^T = D\Psi^* \tag{3.1.26}$$

where $\bar{\Psi} = \Psi^\dagger A$. The chirality condition can be applied to any spinors of even spacetime dimension. For example, in four spacetime dimensions a 4-component spinor is described in terms of two 2-component Weyl spinors.

Under a Lorentz transformation $x^a \to x'^a = \Lambda^a{}_b x^b$, a spinor field transforms as,

$$\Psi(x) \to \Psi'(x') = e^{-\frac{1}{2}\omega_{ab}M^{ab}}\Psi(x) \tag{3.1.27}$$

A spinor field satisfies the Dirac equation,

$$(i\Gamma^a\partial_a - m)\,\Psi(x) = 0 \tag{3.1.28}$$

This is the equation of motion for the action,

$$\mathcal{S} = \int d^Dx\left(-i\bar{\Psi}(x)\Gamma^a\partial_a\Psi(x) + m\bar{\Psi}(x)\Psi(x)\right) \tag{3.1.29}$$

The mass dimension of the spinor fields can be determined from the action.

$$[\Psi] = [\bar{\Psi}] = \frac{1}{2}\,(D-1) \tag{3.1.30}$$

Spacetime dimension	2	4	6	10
Clifford Algebra Dimension	2	4	8	32
Minimal Dimension	1	4	8	16

Table 3.1: Clifford Algebra Dimensions

The dimension of the spinor fields in a higher-dimensional field theory is given by the dimension of the Clifford algebra, summarised in Table 3.1 [3]. The supersymmetry generators are spinors under $SO(D-1,1)$ and therefore representations of the Clifford algebra. The supercharge Q_i and its Dirac conjugate $\bar{Q}_j$ have the anticommutation relation [3],

$$\{Q^A, \bar{Q}_B\} = 2\delta^A{}_B \Gamma^a P_a$$

The dimension of the SUSY generators dictates the internal symmetry group, based on the number of supersymmetries present. As stated in Chapter 2 the internal symmetry group is $U(\mathcal{N})$, with $\mathcal{N}$ labelling the internal symmetry group. The value of $\mathcal{N}$ is given by the (minimal) Clifford algebra dimension n and the total number of supersymmetries m,

$$\mathcal{N} = \frac{m}{n} \tag{3.1.31}$$

For example, a $\mathcal{N} = 1$ SUSY field theory in four dimensions has four supersymmetries, one from each component of the $SO(3,1)$ Clifford algebra representation.

Renormalisation of Higher-Dimensional Field Theories

As was mentioned in the Introduction, the issue of renormalisation is important when studying a theory perturbatively. It is found that a field/gauge theory is non-renormalisable if it contains couplings with a negative mass dimension [1, 4]. Recall the action of a gauge field. If this gauge field is a non-abelian gauge field then,

$$\mathcal{S} = -\frac{1}{4}\int d^D x\, F_{ab}F^{ab} \tag{3.1.32}$$

where the non-abelian field tensor is,

$$F_{ab} = \partial_a A_b - \partial_b A_a + ig[A_a, A_b] \tag{3.1.33}$$

and g is the (Yang-Mills) coupling.[3] From the dimensional analysis of a gauge field, the field tensor has a mass dimension,

$$[F_{ab}] = \frac{1}{2}(D-2) + 1 = \frac{D}{2} \tag{3.1.34}$$

The mass dimension of the coupling is therefore,

$$[g] = \frac{D}{2} - (D-2) = \frac{1}{2}(4-D) \tag{3.1.35}$$

In the four-dimensional spacetime considered in Chapter 2 the coupling is dimensionless, so a pure Yang-Mills theory is renormalisable. However, if $D \geq 5$ then the coupling has a negative mass dimension and the gauge theory is non-renormalisable. Non-renormalisable theories are inconsistent and have limited predictive power. The explicit dependence of 'physical' quantities on the cut-off indicates that the theory is not well-defined in the UV, new physical effects become manifest. The description of the new physics in the UV is provided by a UV completion, a renormalisable gauge theory or string theory with a non-trivial UV fixed point. Little string theory is thought to provide a UV completion to the $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory.

3.2 Phenomenologically Interesting Theories

To study a particular phenomenon, a phenomenological theory must reproduce the properties of that phenomenon, e.g. it must be four-dimensional at the energies that would be observed in an experiment. In order to study a phenomenologically interesting higher-dimensional field theory, the extra dimensions must be resolved with the observation that spacetime is four-dimensional. *Dimensional reduction* is a technique that simply removes the extra dimensions [3, 31]. Consider a higher-dimensional field theory with coordinates x^a, $a = 0, 1, 2, \ldots, D-1$. Dimensional reduction ignores the dependence of a field on the extra dimensions, $\phi(x^a) = \phi(x^\mu)$ for $\mu = 0, 1, 2, 3$. The Lorentz group is reduced e.g. $SO(D-1,1) \to SO(3,1)$ and the higher-dimensional components of a gauge field transform as scalar fields under the new Lorentz group.

[3] In Chapter 2 the gauge fields A_μ were parameterised such that the coupling g_{ym} was an overall coefficient of the field tensor.

Four-dimensional supersymmetric field theories with extended supersymmetry are related to higher-dimensional supersymmetric field theories via dimensional reduction [3]. The number of supersymmetries is preserved by dimensional reduction, so the reduced theory possesses extended supersymmetry. For example, a ten-dimensional SUSY Yang-Mills theory with minimal ($\mathcal{N} = 1$) supersymmetry has sixteen supersymmetries. Applying dimensional reduction to this ten-dimensional theory and reducing the theory to four dimensions produces a four-dimensional SUSY Yang-Mills theory with sixteen supercharges. In four spacetime dimensions a theory with sixteen supercharges has $\mathcal{N} = 4$ supersymmetry, therefore the dimensional reduction of a ten-dimensional $\mathcal{N} = 1$ SUSY Yang-Mills theory to four dimensions is the $\mathcal{N} = 4$ SUSY Yang-Mills theory [43]. The dimensional reduction of the same ten-dimensional theory to six dimensions produces the $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory [43].

As an illustration of the relationship between supersymmetric theories in different dimensions, the dimensional reduction of $U(1)$ $\mathcal{N} = 1$ SUSY Yang-Mills theory in ten dimensions to six dimensions will be demonstrated in order to construct the $U(1)$ $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory [31].[4] The action of the $U(1)$ $\mathcal{N} = 1$ SUSY Yang-Mills in ten dimensions is,

$$\mathcal{S} = \frac{1}{g_{10}^2} \int d^{10}x \, \mathrm{Tr} \left\{ -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} i \bar{\Psi} \Gamma^M D_M \Psi \right\} \tag{3.2.1}$$

where $M = 0, 1, \ldots, 9$. Under dimensional reduction, the ten-dimensional Minkowski spacetime is reduced to a six-dimensional Minkowski spacetime.

$$\Re^{9,1} \longrightarrow \Re^{5,1}$$

Under this reduction the Lorentz group is reduced to a subgroup,

$$SO(9,1) \to SO(5,1) \times SO(4) \tag{3.2.2}$$

$SO(5,1)$ forms the Lorentz group of the six-dimensional theory, whilst $SO(4)$ is the R-symmetry of the theory. The ten-dimensional fields A_M and Ψ become functions of only the remaining six spacetime dimensions x^i, for $i = 0, 1, \ldots, 5$.

$$A_M = A_M(x^i) \qquad \Psi = \Psi(x^i) \tag{3.2.3}$$

[4] In preparation for the Maldacena-Núñez compactification of a gauge theory.

The action of the higher-dimensional differentials on the ten-dimensional fields is,

$$\begin{aligned} \partial_m A_M(x^i) &= 0 \\ \partial_m \Psi(x^i) &= 0 \end{aligned} \qquad m = M - 5 = 1, 2, 3, 4 \tag{3.2.4}$$

The six components of the ten-dimensional gauge field that are longitudinal to the six-dimensional spacetime, transform as a 6-vector under $SO(5,1)$ and form a six-dimensional gauge field A_i. Each of the transverse components of the ten-dimensional gauge field transform as a scalar under $SO(5,1)$ and form four real scalar fields ϕ_m. The ten-dimensional field tensor,

$$F_{MN} = (\partial_M A_N - \partial_N A_M) \tag{3.2.5}$$

is decomposed under dimensional reduction.

$$F_{ij} = (\partial_i A_j - \partial_j A_i) \tag{3.2.6a}$$

$$F_{im} = \partial_i A_m = \partial_i \phi_m = -F_{mi} \tag{3.2.6b}$$

Under dimensional reduction the bosonic action is,

$$\begin{aligned} \mathcal{S}_B &= \frac{1}{g_6^2} \int d^6x \left(-\frac{1}{4} F_{MN} F^{MN} \right) \\ &= \frac{1}{g_6^2} \int d^6x \left(-\frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} \partial_i \phi_m \partial^i \phi^m \right) \end{aligned} \tag{3.2.7}$$

Spinors in a ten-dimensional spacetime are 32-component objects. Under dimensional reduction the spinors of $SO(9,1)$ are decomposed into representations of $SO(5,1) \times SO(4)$. A ten-dimensional theory with minimal supersymmetry (16 supersymmetries) is constructed from 16-component Majorana-Weyl spinors rather than 32-component spinors. The decomposition of the Majorana-Weyl spinors under $SO(9,1) \to SO(5,1) \times SO(4) \sim SO(5,1) \times SU(2)_A \times SU(2)_B$ is,

$$\mathbf{16} \to (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \tag{3.2.8}$$

where $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ are representations of $SU(2)_A \times SU(2)_B$ and $\mathbf{4}$ and $\bar{\mathbf{4}}$ are the fundamental and anti-fundamental representations of $SU(4) \sim SO(5,1)$. The spinor decomposition is induced by the decomposition of the $SO(9,1)$ Clifford algebra,

$$\Gamma^M = \left\{ \tilde{\Gamma}^i \otimes \tilde{\gamma}^5, \mathbb{1}_8 \otimes \tilde{\gamma}^m \right\} \tag{3.2.9}$$

where $\tilde{\Gamma}^i$ is the $SO(5,1)$ Clifford algebra and $\tilde{\gamma}^m$ is the $SO(4)$ Clifford algebra. The Clifford algebra for $SO(5,1)$ is [31],

$$\tilde{\Gamma}^i = \begin{pmatrix} 0 & \Sigma^i \\ \bar{\Sigma}^i & 0 \end{pmatrix} \tag{3.2.10}$$

and the Clifford algebra for $SO(4)$ is [31],

$$\tilde{\gamma}^m = \begin{pmatrix} 0 & \tau^m \\ \bar{\tau}^m & 0 \end{pmatrix} \tag{3.2.11}$$

See Appendix A for further details.

The spinors for the ten-dimensional $\mathcal{N}=1$ theory are Majorana-Weyl spinors of the $SO(9,1)$ Clifford algebra. The first step is to impose the Majorana and chirality conditions on a general 32-component $SO(9,1)$ spinor. The chirality condition for the Clifford algebra (3.2.9),

$$\Psi_\pm = \frac{1}{2}\left(\mathbb{1}_{32} \pm \left(\tilde{\Gamma}^7 \otimes \tilde{\gamma}^5\right)\right)\Psi \tag{3.2.12}$$

decomposes the 32-component spinors of $SO(9,1)$ into Weyl spinors of $SO(5,1)$, with an $SO(4)$ R-symmetry,

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda^A_{\underline{\alpha}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\lambda}^{\underline{\dot{\alpha}}}_A \tag{3.2.13}$$

where $A = 1,2,3,4$ is an $SO(5,1)$ spinor index and $\underline{\alpha}, \underline{\dot{\alpha}} = 1,2$ are $SU(2)_A \times SU(2)_B \sim SO(4)$ spinor indices. The Majorana condition implies the Weyl spinors have the following hermitian conjugates.

$$\left(\lambda^A_{\underline{\alpha}}\right)^\dagger = \bar{\Sigma}^0_{AB}\lambda^{B\underline{\alpha}} \tag{3.2.14a}$$

$$\left(\bar{\lambda}^{\underline{\dot{\alpha}}}_A\right)^\dagger = \Sigma^{0AB}\bar{\lambda}_{A\underline{\dot{\alpha}}} \tag{3.2.14b}$$

The Majorana condition shows that the charge conjugation of a six-dimensional Weyl spinor relates it to itself. Unlike in a four-dimensional theory, charge conjugation does not relate spinors of the left-handed chirality to the right-handed chirality. In a six-dimensional supersymmetric theory, supersymmetry is chiral with the notation

$\mathcal{N} = (A, B)$ denoting the number of supersymmetries for each chirality. The fermionic action is,

$$\begin{aligned}\mathcal{S}_F &= \frac{1}{g_6^2}\int d^6x\left(-\frac{1}{2}\,i\bar{\Psi}\Gamma^M\partial_M\Psi\right)\\ &= \frac{1}{g_6^2}\int d^6x\left(-\frac{1}{2}\,i\lambda^{A\underline{\alpha}}\bar{\Sigma}^i_{AB}\partial_i\lambda^B_{\underline{\alpha}} - \frac{1}{2}\,i\bar{\lambda}_{A\underline{\dot{\alpha}}}\Sigma^{iAB}\partial_i\bar{\lambda}^{\underline{\dot{\alpha}}}_B\right)\end{aligned} \tag{3.2.15}$$

In summary, the full $\mathcal{N} = (1, 1)$ SUSY Yang-Mills action is,

$$\begin{aligned}\mathcal{S} = \frac{1}{g_6^2}\int d^6x\Big(&-\frac{1}{4}\,F_{ij}F^{ij} - \frac{1}{2}\,\partial_i\phi_m\,\partial^i\phi^m\\ &-\frac{1}{2}\,\lambda^{A\underline{\alpha}}\bar{\Sigma}^i_{AB}\partial_i\lambda^B_{\underline{\alpha}} - \frac{1}{2}\,\bar{\lambda}_{A\underline{\dot{\alpha}}}\Sigma^{iAB}\partial_i\bar{\lambda}^{\underline{\dot{\alpha}}}_B\Big)\end{aligned} \tag{3.2.16}$$

Extra dimensions were first proposed by Kaluza [44] in an attempt to unify Maxwell's theory of electromagnetism with Einstein's general theory of relativity. Kaluza considered general relativity in a five-dimensional spacetime and then performed a 4+1 split of the metric. The 4+1 split of the five-dimensional Einstein equation reproduces the four-dimensional Einstein equation along with the Maxwell equation and the Klein-Gordon equation [44]. There were two problems with this proposal, firstly the split of the five-dimensional metric appears unnatural, and secondly a fifth dimension would be physically observable [44]. These problems were resolved by Klein who proposed that the additional dimension was compact and formed a circle. Klein's proposal naturally imposed the 4+1 split of the metric and the compact nature of the fifth dimension would allow its presence to be hidden from some experimental observations. The compact dimension has a finite size $\sim R$, unlike the usual four spacetime dimensions. When probing length scales greater than R, the compact dimension cannot be seen and the universe appears to be four-dimensional. The consequence of the compactification is that momentum in the compact dimension is quantised and the theory possesses an infinite number of massive four-dimensional states, called the Kaluza-Klein modes. Only the zero mode describes Kaluza's theory [44]. Kaluza-Klein theory was the first demonstration of how extra dimensions can exist in a realistic physical theory, the dimensionality of the theory is reduced by compactifying the extra dimensions. The extra dimensions are only manifest at length scales smaller than the size of the compact dimensions.

There are various ways to compactify a theory. Consider the compactification of d dimensions of a $D = 4 + d$-dimensional field theory. The uncompactified higher-dimensional theory has a spacetime manifold $\Re^{3+d,1}$. By deciding to compactify d dimensions of the theory, the full isotropy group of Lorentz transformations $SO(3+d,1)$ must be decomposed to a subgroup,

$$SO(3+d,1) \to SO(3,1) \times SO(d)$$

This decomposition induces a decomposition of the spacetime manifold.

$$\Re^{3+d,1} \to \Re^{3,1} \times \Re^{d}$$

The compactification of the d dimensions involves the replacement of the sub-manifold $\Re^d$ with a compact d-dimensional manifold Σ^d.

$$\Re^d \to \Sigma^d$$

In the classical action, the action on the manifold $\Re^d$ must be replaced with the corresponding action on the curved manifold Σ^d. The construction of a field theory on a curved manifold is discussed in Section 3.3. *Toroidal compactification* is the simplest example of a compactification and is ideal for demonstrating the generic features. (It is not necessary to understand field theory in curved spacetime for the toroidal case). Each dimension is individually compactified to a circle S^1, therefore the compact manifold Σ^d is a higher-dimensional torus. Kaluza-Klein theory is an example of the toroidal compactification of one dimension.

Consider the toroidal compactification of a five-dimensional free scalar field theory,[5]

$$\mathcal{S} = -\int d^5x \; \partial_a\phi\partial^a\phi \tag{3.2.17}$$

for a real scalar field $\phi^\dagger = \phi$. Choosing to compactify the fifth dimension induces a decomposition of the five-dimensional Minkowski spacetime to a sub-manifold,

$$\Re^{4,1} \to \Re^{3,1} \times \Re$$

[5]This calculation is based on a similar calculation in [32].

The induced decomposition of the action is,

$$\mathcal{S} = -\int d^4x \int dy \left(\partial_\mu\phi\partial^\mu\phi + \partial_5\phi\partial^5\phi\right) \tag{3.2.18}$$

where the fifth dimension is now labelled by y. The fifth dimension y is compactified to a circle $\Re \to S^1$ by the identification $y = y + 2\pi R$.

$$\mathcal{S} = -\int d^4x \int_0^{2\pi R} dy \;\left(\partial_\mu\phi(x,y)\partial^\mu\phi(x,y) + \partial_5\phi(x,y)\partial^5\phi(x,y)\right) \tag{3.2.19}$$

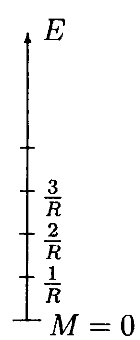
The effective four-dimensional description of the compactified theory is obtained by integrating out the compact dimensions. The fields of the higher-dimensional field theory can be expanded in terms of the eigenstates of the compact manifold. The degrees of freedom of the compact manifold can be integrated-out once they have been separated from the degrees of freedom of the non-compact manifold. In this example, the Fourier expansion of the fields in terms of the eigenstates of the circle is,

$$\phi(x,y) = \sum_{n=-\infty}^{\infty} \phi^{(n)}(x)e^{in\frac{y}{R}} \tag{3.2.20}$$

Momentum on the circle is quantised $p = \frac{n}{R}$, with integer $n = 0, 1, \ldots$ and the infinite number of Fourier coefficients $\phi^{(n)}(x)$ represents the degrees of freedom on the four non-compact dimensions. After performing the Fourier expansion and integrating over the fifth dimension, the effective four-dimensional action is,

$$\mathcal{S} = -2\pi R \int d^4x \sum_{n=-\infty}^{\infty} \left(\partial_\mu\phi^{(n)}(x)\partial^\mu\phi^{(n)}(x) + \left(\frac{n}{R}\right)^2 \phi^{(n)}(x)\phi^{(n)}(x)\right) \tag{3.2.21}$$

The effective four-dimensional theory consists of an infinite tower of massive states, each with a mass $M = \frac{n}{R}$. Each Kaluza-Klein mode is labelled by an integer n.



The first 'excited' Kaluza-Klein mode in the effective four-dimensional theory corresponds to the particles being able to propagate in the compact dimension. Recall that the coupling of a higher-dimensional field theory has a negative mass dimension. The relationship between the higher-dimensional coupling g_d and the four-dimensional coupling g_4 is given by,

$$g_4^2 = \frac{g_d^2}{\mathrm{Vol}\,\Sigma^d} \tag{3.2.22}$$

In the limit $R \to \infty$, the masses of the excited Kaluza-Klein modes become infinite and decouple from the theory.

$$\mathcal{S} = -\int d^4x\ \partial_\mu \phi \partial^\mu \phi \tag{3.2.23}$$

The compactified dimension has been reduced to zero volume $V = 2\pi R = 0$. An equivalent statement is that the probe scale $L \gg R$. The theory is four-dimensional, the four-dimensional fields having been the zero modes of the Kaluza-Klein tower. This limit is dimensional reduction.

3.3 Field Theory in Curved Spacetime

The compactification of a field theory leads to a field theory on a curved spacetime. The study of such theories involves the use of differential geometry, familiar from general relativity, for curved spacetimes. A field theory in curved spacetime must be invariant under general coordinate transformations on the manifold. This invariance is a generalisation of the Lorentz invariance of Minkowski spacetime. A field theory is invariant under general coordinate transformations if its action remain invariant. Furthermore, in the absence of curvature the field theory must reproduce the Minkowski action.

The spin of a field is defined by its representation under the Lorentz group. On a curved spacetime manifold there is no global Lorentz group. However, all curved manifolds are locally flat and a local Lorentz frame can be defined at all points on the manifold, with a local coordinate system ζ^α. The spin of a field on the curved manifold is defined by its transformation under the local Lorentz group.

Under a general coordinate transformation $x^a \to x'^a$, a tensor with contravariant indices $a, b, \ldots$ and covariant indices $c, d, \ldots$ has the following transformation [33].

$$T'^{ab\ldots}_{cd\ldots} = \frac{\partial x'^a}{\partial x^e}\frac{\partial x'^b}{\partial x^f}\frac{\partial x^g}{\partial x'^c}\frac{\partial x^h}{\partial x'^d}\cdots T^{ef\ldots}_{gh\ldots} \tag{3.3.1}$$

All tensors are covariant under general coordinate transformations. The special case is a scalar which is invariant under general coordinate transformations.

The determinant of the metric tensor $g = \det g_{ab}$ does not transform as a tensor under general coordinate transformations. Consider the transformation of the metric tensor under a general coordinate transformation.

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a} g_{cd} \frac{\partial x^d}{\partial x'^b} \tag{3.3.2}$$

This transformation is a matrix equation. The transformation of the determinant of the metric tensor under a general coordinate transformation is given by the determinant of equation (3.3.2).

$$g' = \left|\frac{\partial x}{\partial x'}\right|^2 g \tag{3.3.3}$$

where $\left|\frac{\partial x}{\partial x'}\right|$ is the *Jacobian* of the transformation $x' \to x$. The determinant of the metric g is an example of a *tensor density*, an object which transforms as a tensor except for additional Jacobian factors. Under a general coordinate transformation the volume element transforms as,

$$d^4x' = \left|\left|\frac{\partial x'}{\partial x}\right|\right| d^4x \tag{3.3.4}$$

(the modulus of the inverse Jacobian). An invariant volume element can be constructed using the determinant of the metric,

$$\sqrt{g'}\, d^4x' = \left|\left|\frac{\partial x}{\partial x'}\right|\right| \sqrt{g} \left|\left|\frac{\partial x'}{\partial x}\right|\right| d^4x = \sqrt{g}\, d^4x \tag{3.3.5}$$

A difficulty arises when derivatives are introduced. Derivatives transform as a tensor under a general coordinate transformation.

$$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b} \tag{3.3.6}$$

However a derivative acting on a contravariant vector does not transform as a tensor under a general coordinate transformation.

$$\frac{\partial}{\partial x'^a}V'^b = \frac{\partial x^c}{\partial x'^a}\frac{\partial}{\partial x^c}\left(\frac{\partial x'^b}{\partial x^d}V^d\right) = \frac{\partial x^c}{\partial x'^a}\frac{\partial x'^b}{\partial x^d}\frac{\partial}{\partial x^c}V^d + \frac{\partial x^c}{\partial x'^a}\left(\frac{\partial^2 x'^b}{\partial x^c \partial x^d}\right)V^d \qquad (3.3.7)$$

It is the second term in this expression that prevents the object $\frac{\partial}{\partial x^a}V^b$ from transforming as a tensor under general coordinate transformations. An object which does transform as a tensor under general coordinate transformations can be constructed, the (general) covariant derivative.

$$\nabla_a V^b = \frac{\partial}{\partial x^a}V^b + \Gamma^b_{ac}V^c \qquad (3.3.8)$$

Γ^b_{ac} is called the *Affine connection* or *Christoffel symbol*, and is defined as,

$$\Gamma^b_{ac} = \frac{\partial x^b}{\partial \zeta^\alpha}\frac{\partial^2 \zeta^\alpha}{\partial x^a \partial x^c} \qquad (3.3.9)$$

Like the derivative of a contravariant tensor, the Christoffel symbol does not transform as a tensor under a general coordinate transformation.

$$\Gamma'^b_{ac} = \frac{\partial x'^b}{\partial x^e}\frac{\partial x^d}{\partial x'^a}\frac{\partial x^f}{\partial x'^c}\Gamma^e_{df} - \frac{\partial x^d}{\partial x'^c}\frac{\partial x^e}{\partial x'^a}\frac{\partial^2 x'^b}{\partial x^d \partial x^e} \qquad (3.3.10)$$

If the identity [33],

$$\frac{\partial x'^b}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^a \partial x'^c} = -\frac{\partial x^d}{\partial x'^c}\frac{\partial x^e}{\partial x'^a}\frac{\partial^2 x'^b}{\partial x^d \partial x^e} \qquad (3.3.11)$$

is applied to the transformation of the Christoffel symbol, it is clear that the covariant derivative acting on a contravariant vector acts as a tensor under a general coordinate transformation,

$$\frac{\partial}{\partial x'^a}V'^b + \Gamma'^b_{ac}V'^c = \frac{\partial x'^b}{\partial x^e}\frac{\partial x^d}{\partial x'^a}\left(\frac{\partial}{\partial x^d}V^e + \Gamma^d_{ef}V^f\right) \qquad (3.3.12)$$

In terms of the metric, the Christoffel symbol is,

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) \qquad (3.3.13)$$

A covariant derivative acting on the tensor $T^a_{\ b}$ has the action,

$$\nabla_a T^b_{\ c} = \frac{\partial}{\partial x^a}T^b_{\ c} + \Gamma^b_{ad}T^d_{\ c} - \Gamma^d_{ac}T^b_{\ d} \qquad (3.3.14)$$

The formalism above describes how to construct a field theory of scalar and vector fields in curved spacetime. The treatment is not valid for the description of spinors in curved spacetime. Spinor fields are spinor representations of the Lorentz group, but unlike flat Minkowski/Euclidean spacetime, a curved spacetime has no global Lorentz group. This prevents the definition of a global spinor field [33, 34]. However, at all points in a curved spacetime the manifold is locally flat, there is a local Lorentz frame with a local Lorentz group. The relationship between the local Lorentz coordinates and global coordinates is given by the metric [33, 34],

$$g_{ab}(x) = \left(\frac{\partial \zeta^{\alpha}}{\partial x^{a}}\right)\left(\frac{\partial \zeta^{\beta}}{\partial x^{b}}\right)\eta_{\alpha\beta} \tag{3.3.15}$$

The object $e^{\alpha}_{a} = \left(\frac{\partial \zeta^{\alpha}}{\partial x^{a}}\right)$ is a vielbein[6] and maps the local coordinate system with index α, to the global coordinate system with index a. For example, a local vector field $A_{\alpha}(\zeta)$ is mapped to a global vector field $A_a(x)$ by,

$$A_a(x) = e^{\alpha}_{a}\, A_{\alpha}(\zeta) \tag{3.3.16}$$

When working with a local Lorentz frame the physics must be invariant under local Lorentz transformations rather than the global Lorentz transformations of Minkowski spacetime. A local Lorentz symmetry is a gauge symmetry. As with any gauge symmetry, invariance is ensured under a gauge transformation by the introduction of a gauge field. The gauge field of local Lorentz transformations is the spin connection $R^{\alpha\beta}_{a}$. Consider the object $\partial_a A^{\alpha}$ under a local Lorentz transformation,

$$\partial_a A^{\alpha} \rightarrow \left(\partial_a \Lambda^{\alpha}{}_{\beta}(x)\right) A^{\beta} + \Lambda^{\alpha}{}_{\beta}(x)\partial_a A^{\beta} \tag{3.3.17}$$

If the spin connection is introduced with (gauge) transformation [45],

$$R_{a}{}^{\alpha}{}_{\beta} \rightarrow \Lambda^{\alpha}{}_{\gamma}(x)\, R_{a}{}^{\gamma}{}_{\delta}\left(\Lambda^{-1}(x)\right)^{\delta}{}_{\beta} - \left(\partial_a \Lambda^{\alpha}{}_{\gamma}(x)\right)\left(\Lambda^{-1}(x)\right)^{\gamma}{}_{\beta} \tag{3.3.18}$$

under local Lorentz transformations, then the object,

$$\nabla_a A^{\alpha} = \partial_a A^{\alpha} + R_{a}{}^{\alpha}{}_{\beta} A^{\beta} \tag{3.3.19}$$

[6]The term vielbein is used to denote any dimension. In a particular dimension the dimensionality is given by appropriate number in German, e.g zweibein for two dimensions, dreibein for three.

is Lorentz covariant. The spin connection is given by the following expression [45].

$$\begin{aligned} R_a^{\alpha\beta} &= \frac{1}{2}\, e^{b\alpha}\Big(\partial_a e_b^\beta - \partial_b e_a^\beta\Big) - \frac{1}{2}\, e^{b\beta}\Big(\partial_a e_b^\alpha - \partial_b e_a^\alpha\Big) \\ &\quad -\frac{1}{2}\, e^{b\alpha} e^{c\beta}\Big(\partial_b e_{c\gamma} - \partial_c e_{b\gamma}\Big) e_a^\gamma \end{aligned} \tag{3.3.20}$$

The spin connection is used to treat spinors in curved spacetime. The covariant derivative acting on a spinor is defined as,

$$\nabla_a \psi = \partial_a \psi + \frac{1}{2}\, R_a^{\alpha\beta} M_{\alpha\beta} \psi \tag{3.3.21}$$

where $M_{\alpha\beta}$ is the spin-$\frac{1}{2}$ representation of Lorentz group $SO(D-1,1)$ and the factor of $\frac{1}{2}$ is conventional.

3.4 Spherical Compactification

In Section 3.3 the basic procedure for constructing a field theory on a curved manifold was outlined. In compactifying a theory on a 2-sphere, the field theory on $\Re^2$ is replaced with the corresponding theory on S^2. The spin of the fields is given by their transformation properties under the local 'Lorentz' group, $SO(2)$. This Section begins by describing the 2-sphere and its group structure before proceeding to study the eigenstates of the 2-sphere and their corresponding field theory.

3.4.1 The 2-Sphere and Group Structure

The 2-sphere is a two-dimensional manifold with coordinate system $q_a = (\theta, \phi)$. It is constructed by embedding the manifold in $\Re^3$ through the defining equation,

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{3.4.1}$$

with a three-dimensional coordinate system x_i in $\Re^3$. The 2-sphere is the surface/boundary of a three-dimensional ball (a 3-ball). The coordinate basis x_i is defined in terms of the coordinates on the 2-sphere and the radius R from the origin of the 3-ball to its boundary [35].

$$x_1 = R \sin\theta \cos\phi \tag{3.4.2a}$$
$$x_2 = R \sin\theta \sin\phi \tag{3.4.2b}$$
$$x_3 = R \cos\theta \tag{3.4.2c}$$

This dictates the metric of the 2-sphere.

$$\begin{aligned} ds^2 &= R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2 \\ &= g_{ab}\, dq^a dq^b \end{aligned} \tag{3.4.3}$$

The isometries of the 2-sphere are described by the group of rotations in three dimensions $SO(3) \sim SU(2)$. $SO(3)$ is analogous to the Poincaré group in Minkowski spacetime. $SO(3) \sim SU(2)$ has the Lie algebra [35],

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \tag{3.4.4}$$

whose generators (in the defining representation) are the orbital angular momentum operators L_i [35].

$$L_i = -i\varepsilon_{ijk} x_j \partial_k \tag{3.4.5}$$

In terms of coordinates on the 2-sphere the orbital angular momentum operators are [35],

$$L_1 = i \sin\phi \frac{\partial}{\partial\theta} + i \cos\phi \cot\theta \frac{\partial}{\partial\phi} \tag{3.4.6a}$$
$$L_2 = -i \cos\phi \frac{\partial}{\partial\theta} + i \sin\phi \cot\theta \frac{\partial}{\partial\phi} \tag{3.4.6b}$$
$$L_3 = -i \frac{\partial}{\partial\phi} \tag{3.4.6c}$$

These expressions can be summarised as [23],

$$L_i = -iK_i^a \partial_a \tag{3.4.7}$$

The metric tensor can be expressed in terms of these Killing vectors K_i^a,

$$g^{ab} = \frac{1}{R^2} K_i^a K_i^b \tag{3.4.8}$$

The Lie groups $SO(3)$ and $SU(2)$ are locally isomorphic.[7] For a generator J_i, $i = 1, 2, 3$, Lie algebra of $SO(3)$ and $SU(2)$ is,

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \tag{3.4.9}$$

The Casmir operator is $J^2 = J_iJ_i$. The eigenstates of $SU(2)$ are simultaneous eigenstates of J^2 and J_3, labelled $|j, m>$. Under the action of these operators,

$$J^2|j, m> \;=\; j(j+1)|j, m> \tag{3.4.10a}$$

$$J_3|j, m> \;=\; m|j, m> \tag{3.4.10b}$$

The quantum numbers are $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m = -j, -j+1, \ldots, j$. The quantum numbers j label the representations of $SU(2)$, with each representation having a dimension $N = 2j + 1$. Furthermore, the operators $J_\pm = J_1 \pm iJ_2$ can be defined which have the commutation relations,

$$[J^2, J_\pm] \;=\; 0 \tag{3.4.11a}$$

$$[J_3, J_\pm] \;=\; \pm J_\pm \tag{3.4.11b}$$

Their action on the eigenstates is,

$$J_\pm|jm>= \sqrt{(j \pm m + 1)(j \mp m)}|jm \pm 1>= j_\pm|jm \pm 1> \tag{3.4.12}$$

The full spectrum of eigenstates of $SU(2)$ is derived by repeated application of the lowering operator J_- to the highest weighted state, for a given j [36].

[7] An isomorphism is a 1:1 mapping from one group to another that preserves the group multiplication [36]. $SO(3) \sim SU(2)$ is a 2:1 mapping, however the Lie algebras of the two groups are isomorphic, therefore the two groups are locally isomorphic.

The matix representations of the $SU(2)$ generators are defined below.

$$\begin{aligned} J_1^{(N)} =< j,m|J_1|j',m' > &= \frac{1}{2}\left(j'_+\delta_{ll'}\delta_{m,m'+1} + j'_-\delta_{ll'}\delta_{m,m'-1}\right) \\ &= \frac{1}{2}\left(j_-\delta_{ll'}\delta_{m,m'+1} + j_+\delta_{ll'}\delta_{m,m'-1}\right) \end{aligned} \tag{3.4.13}$$

$$\begin{aligned} J_2^{(N)} =< j,m|J_2|j',m' > &= -\frac{i}{2}\left(j'_+\delta_{ll'}\delta_{m,m'+1} - j'_-\delta_{ll'}\delta_{m,m'-1}\right) \\ &= -\frac{i}{2}\left(j_-\delta_{ll'}\delta_{m,m'+1} - j_+\delta_{ll'}\delta_{m,m'-1}\right) \end{aligned} \tag{3.4.14}$$

$$J_3^{(N)} = < j,m|J_3|j',m' > = m\,\delta_{ll'}\delta_{mm'} \tag{3.4.15}$$

$$J^2_{(N)} = < j,m|J^2|j',m' > = j(j+1)\,\delta_{ll'}\delta_{mm'} \tag{3.4.16}$$

Each matrix has dimension $N = 2j+1$. The treatment above for $SU(2)$ can be applied to both orbital and spin angular momentum. Orbital angular momentum has integer quantum number l only, $l = 0,1,2,\dots$, and eigenstates $|l,m>$. Spin angular momentum has quantum number $s = 0,\frac{1}{2},1,\frac{3}{2},2,\dots$, and eigenstates $|s,m_s>$.

3.4.2 Scalar Fields on the 2-Sphere

A scalar field on the 2-sphere is a function of the coordinates q^a defined on the 2-sphere and is invariant under a $SU(2)$ transformation. Any scalar field can be expanded in the scalar eigenstates of the 2-sphere, the eigenstates of orbital angular momentum. The eigenstates of orbital angular momentum in the coordinate basis q^a are the *spherical harmonics* [35],

$$Y_{lm}(\theta,\phi) =< q|l,m > \tag{3.4.17}$$

Under the action of the operators L_i the spherical harmonics have the same properties as (3.4.4).

$$L^2\,Y_{lm}(\theta,\phi) = l(l+1)\,Y_{lm}(\theta,\phi) \tag{3.4.18a}$$

$$L_3\,Y_{lm}(\theta,\phi) = m\,Y_{lm}(\theta,\phi) \tag{3.4.18b}$$

The functional form of the spherical harmonics is derived by solving equations (3.4.18). The separatation of variables $Y_{lm}(\theta,\phi) = \Phi(\phi)\Theta(\theta)$ allows $\Phi(\phi)$ and $\Theta(\theta)$ to be solved separately. $\Phi(\phi) = e^{im\phi}$ for integer m, whilst $L^2\Theta(\theta)$ is solved by a Legendre polynomial [35]. The orthogonality condition for the spherical harmonics is,

$$\int d\Omega\, Y_{lm}^{\dagger} Y_{l'm'} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\; Y_{lm}^{\dagger} Y_{l'm'} = \delta_{ll'}\,\delta_{mm'} \tag{3.4.19}$$

The complex conjugation of a spherical harmonic is,

$$Y_{lm}^{*}(\theta,\phi) = (-1)^m Y_{l,-m}(\theta,\phi) \tag{3.4.20}$$

A scalar field $a(\theta,\phi)$ on the 2-sphere can be expanded in spherical harmonics.

$$a(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi) \tag{3.4.21}$$

where a_{lm} is a complex coefficient. If the scalar a is real, i.e. $a^*(\theta,\phi) = a(\theta,\phi)$, then the complex coefficient satisfies,

$$(a_{lm})^* = \bar{a}_{lm} = (-1)^m a_{l,-m} \tag{3.4.22}$$

The action of a free complex scalar field of mass m on a 2-plane is,

$$\mathcal{S}_A = \int d^2x \Big\{ -\partial_a A^{\dagger}(x)\,\partial^a A(x) + m^2 A^{\dagger}(x) A(x) \Big\} \tag{3.4.23}$$

The compactification of a flat spacetime to a curved spacetime is outlined in Sections 3.2 and 3.3. Compactifying the action (3.4.23) on a 2-sphere, the coordinates of the 2-plane are replaced with the coordinates of the 2-sphere.

$$x^a = \{x^1, x^2\} \rightarrow q^a = \{\theta, \phi\} \tag{3.4.24}$$

The Euclidean metric of the 2-plane is replaced with the curved metric of the 2-sphere,

$$\eta^{ab} = \delta^{ab} \rightarrow g^{ab} \tag{3.4.25}$$

and the volume element is replaced with the invariant volume element,

$$d^2x \rightarrow d^2q\sqrt{g} = R^2 d\theta d\phi \sin\theta = R^2 d\Omega \tag{3.4.26}$$

Partial derivatives on the 2-plane are replaced with generally covariant derivatives on the 2-sphere. A general covariant derivative acting on a scalar is simply a partial derivative.

$$\partial_a A(x) \to \nabla_a A(\theta, \phi) = \partial_a A(\theta, \phi) \tag{3.4.27}$$

The action of a scalar field on a 2-sphere is therefore,

$$\begin{aligned} S_A &= \int R^2 d\Omega \Big\{ -\partial_a A^\dagger(\theta,\phi)\, \partial^a A(\theta,\phi) + m^2 A^\dagger(\theta,\phi) A(\theta,\phi) \Big\} \\ &= \int R^2 d\Omega \Big\{ A^\dagger(\theta,\phi) \Delta_{S^2} A(\theta,\phi) + m^2 A^\dagger(\theta,\phi) A(\theta,\phi) \Big\} \end{aligned} \tag{3.4.28}$$

where Δ_{S^2} is the scalar Laplacian on the 2-sphere,

$$\begin{aligned} \Delta_{S^2} &= \frac{1}{\sqrt{g}}\, \partial_a (g^{ab} \sqrt{g}\, \partial_b) \\ &= \frac{1}{R^2} \left(\csc\theta \partial_\theta (\sin\theta \partial_\theta) + \csc^2\theta \partial_\phi \partial_\phi \right) \\ &= -\frac{1}{R^2} L^2 \end{aligned} \tag{3.4.29}$$

The scalar Laplacian has the following eigenvalues,

$$\Delta_{S^2} Y_{lm}(\theta,\phi) = -\frac{1}{R^2}\, l(l+1) Y_{lm}(\theta,\phi) \tag{3.4.30}$$

with a degeneracy of $2l+1$.

3.4.3 Spinor Fields on the 2-Sphere

A spinor field on the 2-sphere is a spinor representation of $SU(2)$, transforming under the spin-$\frac{1}{2}$ representation. In the spin-$\frac{1}{2}$ matrix representation, the spin operators are defined by the 2×2 Pauli matrices $S_i = \frac{1}{2}\sigma_i$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.4.31}$$

The spin-$\frac{1}{2}$ eigenstates of spin operators S^2 and S_3 are,

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.32}$$

The eigenstates of spin-$\frac{1}{2}$ particles on the 2-sphere are the *spherical spinors.* They are eigenstates of the total angular momentum, $\hat{L}^2$ and $\hat{L}_3$. Spherical spinors are constructed from the spherical harmonics and the spin-$\frac{1}{2}$ eigenstates of spin operators S^2 and S_3. This involves combining two distinct systems, each with a distinct vector space.

If there are two commuting sets of angular momentum operators $(J_1)_i$ and $(J_2)_i$, with eigenstates $|j_1, m_1>$ and $|j_2, m_2>$, then a product basis can be defined as [35, 37],

$$|j_1, j_2, m_1, m_2>=|j_1, m_1> \otimes |j_2, m_2>=|j_1, m_1>|j_2, m_2> \tag{3.4.33}$$

A new basis can be constructed from this product basis. Let the eigenstates of the new basis be $|jm>$ with angular momentum operators J_i. The operators $(J_1)_i$ and $(J_2)_i$ both commute with J_i^2 and J_3, which allows $|j,m>$ to be a simultaneous eigenstate of J_i^2, J_3, $(J_1)_i$ and $(J_2)_i$, therefore,

$$|j,m>=\sum_{m_1,m_2} C(j_1,j_2,j;m_1,m_2,m)|j_1,m_1>|j_2,m_2> \tag{3.4.34}$$

The functions $C(j_1,j_2,j;m_1,m_2,m)$ are called a *Clebsch-Gordan coefficients.*

The spherical spinors are [37],

$$\Omega_{jlm}(\theta,\phi)=\sum_{\mu} C(l,\frac{1}{2},j;m-\mu,\mu,m)Y_{l,m-\mu}(\theta,\phi)\chi_{\mu} \tag{3.4.35}$$

where the Clebsch-Gordan coefficients are given in Table 3.2. j and m are quantum numbers of the total angular momentum. From equation (3.4.35), the spherical

	$m_s=\frac{1}{2}$	$m_s=-\frac{1}{2}$
$j=l+\frac{1}{2}$	$\sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}$	$\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$
$j=l-\frac{1}{2}$	$-\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$	$\sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}$

Table 3.2: $C(l,\frac{1}{2},j;m-m_s,m_s,m)$

spinors are,

$$\Omega^{\hat{\alpha}}_{l+\frac{1}{2},lm}(\theta,\phi) = \begin{pmatrix} \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\, Y_{l,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}\, Y_{l,m+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{3.4.36a}$$

$$\Omega^{\hat{\alpha}}_{l-\frac{1}{2},lm}(\theta,\phi) = \begin{pmatrix} -\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}\, Y_{l,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\, Y_{l,m+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{3.4.36b}$$

where $\hat{\alpha}$ is a spinor index labelling the two components. The spherical spinors are eigenstates of the Dirac operator on the 2-sphere [37],

$$\kappa = -\frac{1}{R}\left(\mathbb{1} + \sigma_i L_i\right) \tag{3.4.37}$$

The presence of the (orbital) angular momentum operators L_i in this Dirac operator indicates that the operator is in a cartesian basis of the three-dimensional Euclidean space. The action of the Dirac operator on the spherical spinors is,

$$\kappa^{\hat{\alpha}}{}_{\hat{\beta}}\, \Omega^{\hat{\beta}}_{q_\pm lm}(\theta,\phi) = \frac{1}{R}\,\kappa_\pm\, \Omega^{\hat{\alpha}}_{q_\pm lm}(\theta,\phi) \tag{3.4.38}$$

The object $\kappa_\pm = \mp(q_\pm + \frac{1}{2})$, with the quantum number $q_\pm = l \pm \frac{1}{2}$ of the total angular momentum. In analogy to the spherical harmonics, a spinor field on the 2-sphere can be expanded in terms of the spherical spinors,

$$\Psi^{\hat{\alpha}}(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-q_\pm}^{q_\pm} \left\{ \psi^{(+)}_{lm} \Omega^{\hat{\alpha}}_{l+\frac{1}{2},lm}(\theta,\phi) + \psi^{(-)}_{lm} \Omega^{\hat{\alpha}}_{l-\frac{1}{2},lm}(\theta,\phi) \right\} \tag{3.4.39}$$

where $\psi^{(\pm)}_{lm}$ are complex coefficients. The orthogonality condition of the spherical spinors is inherited from the spherical harmonics,

$$\int d\Omega\, \Omega^{\dagger}_{qm\hat{\alpha}}(\theta,\phi)\, \Omega^{\hat{\alpha}}_{q'm'}(\theta,\phi) = \delta_{qq'}\delta_{mm'} \tag{3.4.40}$$

The action of a free spinor field of mass m on a 2-plane is,

$$\mathcal{S}_\Upsilon = \int d^2x \left\{ \bar{\Upsilon}(x)(-i\hat{\gamma}^a \partial_a - m)\Upsilon(x) \right\} \tag{3.4.41}$$

The Clifford algebra on the 2-plane is $\hat{\gamma}^a = \left(\sigma^1, \sigma^2\right)$ and the object $\hat{\gamma}^a\partial_a$ is the usual Dirac operator in a flat two-dimensional space. To compactify this free spinor field on the 2-sphere, define the zweibein of the 2-sphere.

$$e^\alpha_a = \mathrm{diag}(R, R\sin\theta) \qquad g_{ab} = \delta_{\alpha\beta}\, e^\alpha_a\, e^\beta_b \tag{3.4.42}$$

The coordinates on the 2-sphere are denoted by the index a, b and the local frame by α, β. The local 'Lorentz' group is $SO(2)$, whose Clifford algebra $\hat{\gamma}^\alpha = \{\sigma^1, \sigma^2\}$ forms a spinor representation,

$$\{\hat{\gamma}^\alpha, \hat{\gamma}^\beta\} = 2\,\delta^{\alpha\beta} \tag{3.4.43a}$$

$$M^{\alpha\beta} = -\frac{i}{4}\,[\hat{\gamma}^\alpha, \hat{\gamma}^\beta] \tag{3.4.43b}$$

The partial derivative on the 2-plane is replaced with the generally covariant derivative on the 2-sphere.

$$\partial_a \Upsilon(x) \rightarrow \nabla_a \Upsilon(\theta, \phi) = \partial_a \Upsilon(\theta, \phi) + \frac{i}{2}\, R_a^{\alpha\beta} M_{\alpha\beta} \Upsilon(\theta, \phi) \tag{3.4.44}$$

The spin connection is calculated from equation (3.3.20), the non-zero components on the 2-sphere are,

$$R_\phi^{12} = -R_\phi^{21} = -\cos\theta \tag{3.4.45}$$

The generators $M_{\alpha\beta}$ of the spin-$\frac{1}{2}$ representation of $SO(2)$ are,

$$M_{12} = -M_{21} = -\frac{i}{4}[\hat{\gamma}_1, \hat{\gamma}_2] = \frac{1}{2}\,\sigma_3 \tag{3.4.46}$$

The Dirac operator on the 2-sphere is defined as,

$$-i\hat{\nabla}_{S^2} = -ie^{a\alpha}\hat{\gamma}_\alpha \nabla_a \tag{3.4.47}$$

In summary, the Dirac operator on the 2-sphere is,

$$-i\hat{\nabla}_{S^2} = -\frac{i\sigma_1}{R}\left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2}\right) - \frac{i\sigma_2}{R\sin\theta}\frac{\partial}{\partial\phi} \tag{3.4.48}$$

and the action of a free spinor field on the 2-sphere is,

$$\mathcal{S}_\Upsilon = \int R^2 d\Omega \left\{\bar{\Upsilon}(\theta, \phi)(-i\hat{\nabla}_{S^2} - m)\Upsilon(\theta, \phi)\right\} \tag{3.4.49}$$

Notice that the Dirac operator $-i\hat{\nabla}_{S^2}$ is not the same operator as the Dirac operator κ of the spherical spinors Ω_{jlm}. Both are Dirac operators on the 2-sphere, but different irreducible representations, with $-i\hat{\nabla}_{S^2}$ as an operator in the spherical basis $q^a = (\theta, \phi)$ and κ as an operator in the cartesian basis $x_i = (x_1, x_2, x_3)$. The Dirac operator $-i\hat{\nabla}_{S^2}$ has a different set of eigenspinors Υ_{jm} to the Dirac operator κ [38].

The Dirac operator has eigenvalues μ and eigenspinors Υ_{jm} [38].

$$-i\hat{\nabla}_{S^2}\Upsilon_{jm} = \mu\Upsilon_{jm} \tag{3.4.50}$$

The eigenvalues and eigenspinors are obtained by solving this equation [38]. The eigenspinor is subject to a separation of variables, with the ϕ dependence solved by a plane wave and the θ dependence is solved by a Jacobi polynomial [38]. The eigenspinors are eigenstates of the total angular momentum, in the spin-$\frac{1}{2}$ representation. The $SU(2)$ generators of this representation are,

$$\hat{L}_3 = -i\partial_\phi \tag{3.4.51a}$$

$$\hat{L}_\pm = \pm e^{\pm i\phi}\left(\partial_\theta \pm i\cot\theta\,\partial_\phi \pm \frac{1}{2}\sin\theta\,\sigma_3\right) \tag{3.4.51b}$$

The action of these operators on the eigenspinors is,

$$\hat{L}_3\Upsilon_{jm} = m\Upsilon_{jm} \tag{3.4.52a}$$

$$\hat{L}_\pm\Upsilon_{jm} = \sqrt{(j\pm m+1)(j\mp m)}\,\Upsilon_{jm\pm1} \tag{3.4.52b}$$

The Casmir operator of the total angular momentum is $\hat{L}^2$.

$$\hat{L}^2 = -\cot\theta\,\partial_\theta - \partial_\theta\partial_\theta - \csc^2\theta\,\partial_\phi\partial_\phi + i\sigma_3\csc\theta\cot\theta\,\partial_\phi + \frac{1}{4}\csc^2\theta \tag{3.4.53}$$

The action of the Casmir operator on the eigenspinors Υ_{jm} is,

$$\hat{L}^2\Upsilon_{jm} = j(j+1)\Upsilon_{jm} \tag{3.4.54}$$

where the quantum number $j = \frac{1}{2}, \frac{3}{2}, \ldots$ with degeneracy $2j+1$. The square of the Dirac operator in the spherical basis satisfies,

$$\left(-i\hat{\nabla}_{S^2}\right)^2\Upsilon_{jm} = \mu^2\Upsilon_{jm} \tag{3.4.55}$$

where the Dirac operator is,

$$\begin{aligned}(-i\hat{\nabla}_{S^2})^2 = -\frac{1}{R^2}\Big(&\cot\theta\,\partial_\theta + \partial_\theta\partial_\theta + \csc^2\theta\,\partial_\phi\partial_\phi \\ &- i\sigma_3\csc\theta\cot\theta\,\partial_\phi - \frac{1}{4} - \frac{1}{4}\csc^2\theta\Big)\end{aligned} \tag{3.4.56}$$

There is a relationship between the Casmir operator $\hat{L}^2$ and the square of the Dirac operator.

$$(-i\hat{\nabla}_{S^2})^2 = \frac{1}{R^2}\left(\hat{L}^2 + \frac{1}{4}\right) \tag{3.4.57}$$

Using the Casmir operator the eigenvalues of the Dirac operator can be determined.

$$\begin{aligned} \left(-i\hat{\nabla}_{S^2}\right)^2 \Upsilon_{jm} &= \mu^2 \Upsilon_{jm} \\ \frac{1}{R^2}\left(\hat{L}^2 + \frac{1}{4}\right)\Upsilon_{jm} &= \frac{1}{R^2}\left(j(j+1) + \frac{1}{4}\right)\Upsilon_{jm} \\ &= \frac{1}{R^2}\left(j + \frac{1}{2}\right)^2 \Upsilon_{jm} \end{aligned} \tag{3.4.58}$$

The eigenvalue of the squared Dirac operator is $\mu^2 \sim \left(j + \frac{1}{2}\right)^2$ for half-integer (total) angular momentum quantum number $j = \frac{1}{2}, \frac{3}{2}, \ldots$, with degeneracy $2j+1$. Hence the Dirac operator has eigenvalue $\mu \sim \pm\left(j + \frac{1}{2}\right)$, each with a corresponding eigenspinor $\Upsilon^{\pm}_{jm}$. The orthogonality condition of the spherical spinors Υ^{ϵ}_{jm} is,

$$\int d\Omega \left(\Upsilon^{\epsilon}_{jm}\right)^{\dagger} \Upsilon^{\epsilon'}_{j'm'} = \delta^{\epsilon\epsilon'}\delta_{jj'}\delta_{mm'} \tag{3.4.59}$$

where $\epsilon, \epsilon' = \pm$.

There are two complete orthonormal sets of spinors on the 2-sphere. The first set of spherical spinors Ω are eigenspinors of the Dirac operator κ, where L_i are the cartesian orbital angular momentum operators of the embedding space. The second set of spherical spinors Υ are eigenspinors of the Dirac operator $-i\nabla_{S^2} = -ie^{a\alpha}\hat{\gamma}_\alpha\nabla_a$, where a labels the coordinates (θ, ϕ) of the 2-sphere. Both κ and $-i\nabla_{S^2}$ must be different representations of the Dirac operator on the 2-sphere, related via a similarity transformation [38]. For two spinors, one in the cartesian basis $\psi(x)$ and one in the spherical basis $\psi(q)$, the spinor transformation between the two bases is [38],

$$\psi^{\hat{\alpha}}(x) = (V^{\dagger})^{\hat{\alpha}}{}_{\hat{\beta}}\psi^{\hat{\beta}}(q) \tag{3.4.60}$$

with the unitary matrix V,

$$V = e^{\frac{i}{2}\sigma_2\theta}e^{\frac{i}{2}\sigma_3\phi} = \begin{pmatrix} e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ -e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{3.4.61}$$

The Dirac operators are related by the similarity transformation,

$$(\hat{\gamma}_3)^{\hat{\alpha}}{}_{\hat{\gamma}}(-i\hat{\nabla})^{\hat{\gamma}}{}_{\hat{\beta}} = V^{\hat{\alpha}}_{\ \hat{\gamma}}\, \kappa^{\hat{\gamma}}{}_{\hat{\delta}}\, (V^{\dagger})^{\hat{\delta}}{}_{\hat{\beta}} \tag{3.4.62}$$

where $\hat{\gamma}_3 = \hat{\gamma}_1\hat{\gamma}_2$ is the two-dimensional chirality operator. Applying the unitary operator $V^{\dagger}$ to the spherical spinors Υ,

$$V^{\dagger}\Upsilon^{\pm}_{jm} = \frac{1}{\sqrt{2}}\left(\Omega_{j,j-\frac{1}{2},m} \mp \Omega_{j,j+\frac{1}{2},m}\right) \tag{3.4.63}$$

Both bases of spherical spinors diagonalise two operators [38], both diagonalise the total angular momentum $\hat{L}^2$ (by definition), the spherical spinor Υ diagonalises the Dirac operator on the 2-sphere $(-i\hat{\nabla}_{S^2})^2$ whilst the spherical spinor Ω diagonalises the orbital angular momentum L^2.

3.4.4 Vector Fields on the 2-Sphere

A vector field on the 2-sphere is a spin-1 representation of the total angular momentum $SU(2)$. All renormalisable quantum field theories with vector particles must be gauge theories, therefore this Section will discuss gauge fields. Eigenstates of the total angular momentum for spin-1 particles, the vector harmonics, must be formed from the spherical harmonics and the spin-1 eigenstates of S^2 and S_3. The spin-1 matrix representation of the spin operators is given by the general formulae (3.4.13) - (3.4.16), however a more useful basis for calculating the eigenstates is [37],

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4.64}$$

These matrices gives the eigenstates,

$$\vec{\xi}_1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \vec{\xi}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{\xi}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \tag{3.4.65}$$

The vector harmonics are [37],

$$\vec{Y}_{jlm}(\theta,\phi) = \sum_{\sigma} C(l,1,j;m-\sigma,\sigma,m)Y_{l,m-\sigma}(\theta,\phi)\vec{\xi}_{\sigma} \tag{3.4.66}$$

where j, l, m are integer quantum numbers. The Clebsch-Gordan coefficients are given in Table 3.3.

	$m_s = 1$	$m_s = 0$	$m_s = -1$
$j = l+1$	$\sqrt{\frac{(l+m)(l+m+1)}{2(2l+1)(l+1)}}$	$\sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(l+1)}}$	$\sqrt{\frac{(l-m)(l-m+1)}{2(2l+1)(l+1)}}$
$j = l$	$-\sqrt{\frac{(l+m)(l-m+1)}{2l(l+1)}}$	$\frac{m}{\sqrt{l(l+1)}}$	$\sqrt{\frac{(l-m)(l+m+1)}{2l(l+1)}}$
$j = l-1$	$\sqrt{\frac{(l-m)(l-m+1)}{2l(2l+1)}}$	$-\sqrt{\frac{(l-m)(l+m)}{l(2l+1)}}$	$\sqrt{\frac{(l+m)(l+m+1)}{2l(2l+1)}}$

Table 3.3: $\mathrm{C}(l, 1, j; m - m_s, m_s, m)$

This Thesis will use a basis which takes advantage of the $SO(2)$ 'Lorentz' symmetry [37, 39],

$$\begin{aligned} \vec{T}_{jm} &= \vec{Y}_{jjm}(\theta,\phi) = \frac{1}{\sqrt{j(j+1)}}\left[\frac{\partial Y_{jm}}{\partial\theta}\vec{\phi} - \csc\theta\frac{\partial Y_{jm}}{\partial\phi}\vec{\theta}\right] && (3.4.67a) \\ &= \frac{1}{\sqrt{j(j+1)}}\, i\,\vec{L}\,Y_{jm} \\ \vec{S}_{jm} &= \sqrt{\frac{j+1}{2j+1}}\,\vec{Y}_{jj-1,m}(\theta,\phi) + \sqrt{\frac{j}{2j+1}}\,\vec{Y}_{jj+1,m}(\theta,\phi) && (3.4.67b) \\ &= \frac{1}{\sqrt{j(j+1)}}\left[\frac{\partial Y_{jm}}{\partial\theta}\vec{\theta} + \csc\theta\frac{\partial Y_{jm}}{\partial\phi}\vec{\phi}\right] = \frac{1}{\sqrt{j(j+1)}}\vec{\partial}\,Y_{jm} \\ \vec{R}_{jm} &= \sqrt{\frac{j}{2j+1}}\,\vec{Y}_{jj-1,m}(\theta,\phi) - \sqrt{\frac{j+1}{2j+1}}\,\vec{Y}_{jj+1,m}(\theta,\phi) && (3.4.67c) \\ &= \vec{r}\,Y_{jm} \end{aligned}$$

where $\vec{\theta}$, $\vec{\phi}$ and $\vec{r}$ are unit vectors in the θ, ϕ and radial directions of the 2-sphere, respectively. The harmonics $\vec{T}_{jm}$ and $\vec{S}_{jm}$ are tangential to the 2-sphere whilst $\vec{R}_{jm}$ is normal to the 2-sphere. By restricting to vectors on a unit 2-sphere, the radial unit vector is $\vec{r} = 0$ and therefore the vector harmonic $\vec{R}_{jm} = 0$. The vector harmonics (3.4.67) are described in terms of the ordinary vectors common to non-relativistic vector analysis. Relativistic theories are described in terms of contravariant and covariant vectors; see Appendix B for their relationship to ordinary vectors. The ordinary vectors can be converted to covariant and contravariant vectors on the 2-sphere by the identity [33],

$$\vec{V}_i = h_i V^i = h_i^{-1} V_i \tag{3.4.68}$$

where $\vec{V}_i$ denotes an ordinary vector and $g_{ij} = h_i^2 \delta_{ij}$. The corresponding covariant vector harmonics are,

$$\begin{aligned} T_{jm\,a} &= \frac{1}{\sqrt{j(j+1)}} R \left[\sin\theta\, \partial_\theta Y_{jm}\, \hat{\phi} - \csc\theta\, \partial_\phi Y_{jm}\, \hat{\theta}\right] && (3.4.69a) \\ S_{jm\,a} &= \frac{1}{\sqrt{j(j+1)}} R \left[\partial_\theta Y_{jm}\, \hat{\theta} + \partial_\phi Y_{jm}\, \hat{\phi}\right] && (3.4.69b) \end{aligned}$$

where the spacetime index $a = \theta, \phi$.

The action of a gauge field on a flat two-dimensional space is,

$$\mathcal{S}_n = -\frac{1}{4} \int d^2x\, F_{ab}(x) F^{ab}(x) \tag{3.4.70}$$

where $F_{ab}(x) = \partial_a n_b(x) - \partial_b n_a(x)$ is the field tensor for a gauge field $n_a(x)$. Compactifying on the 2-sphere, partial derivatives are replaced with generally covariant derivatives,

$$\partial_a n_b(x) \to \nabla_a n_b(\theta, \phi) = \partial_a n_b(\theta, \phi) + \Gamma^c_{ab} n_c(\theta, \phi) \tag{3.4.71}$$

On the 2-sphere the field tensor becomes,[8]

$$\begin{aligned} F_{ab} \to \mathcal{F}_{ab} &= \partial_a n_b(\theta, \phi) + \Gamma^c_{ab} n_c(\theta, \phi) - \partial_b n_a(\theta, \phi) - \Gamma^c_{ba} n_c(\theta, \phi) \\ &= \partial_a n_b(\theta, \phi) - \partial_b n_a(\theta, \phi) && (3.4.72) \end{aligned}$$

and consequently the action of a gauge field on the 2-sphere is,

$$\mathcal{S}_n = -\frac{1}{4} \int R^2 d\Omega\, \mathcal{F}_{ab}(\theta, \phi) \mathcal{F}^{ab}(\theta, \phi) \tag{3.4.73}$$

A Maxwell field on a 2-sphere is a $U(1)$ gauge field with the gauge transformation,

$$n_a \to n_a - R\, \partial_a \chi \tag{3.4.74}$$

[8]The use of $\mathcal{F}_{ab}$ instead of F_{ab} for the field tensor in curved space is introduced to help distinguish between a four-dimensional field tensor and a two-dimensional field tensor in later calculations.

If the gauge field on the 2-sphere is expanded in vector harmonics and the scalar field χ is expanded in spherical harmonics then under a gauge transformation the components of (3.4.74) transform as,

$$\begin{aligned} n'_\theta(\theta,\phi) &= R\sum_{jm}\left(t_{jm}\left(-\csc\theta\right)\partial_\phi Y_{jm} + s_{jm}\partial_\theta Y_{jm} - \chi_{jm}\partial_\theta Y_{jm}\right) \\ n'_\phi(\theta,\phi) &= R\sum_{jm}\left(t_{jm}\sin\theta\,\partial_\theta Y_{jm} + s_{jm}\,\partial_\phi Y_{jm} - \chi_{jm}\,\partial_\phi Y_{jm}\right) \end{aligned}$$

where t_{jm}, s_{jm} and χ_{jm} are the complex coefficients associated with the vector harmonics $\vec{T}_{jm}$, $\vec{S}_{jm}$ and the spherical harmonic Y_{jm}. It follows that the complex coefficient can be set to zero $s_{jm} = 0$ via a gauge transformation with $\chi_{jm} = s_{jm}$. The corresponding gauge fixing condition is the generally covariant analogue of the Lorentz gauge.

$$\begin{aligned} \nabla^a n_a &= g^{ab}\nabla_b n_a \\ &= g^{ab}\partial_b n_a - g^{ab}\Gamma^c_{ba} n_c \end{aligned} \tag{3.4.75}$$

For the 2-sphere there are only three non-zero Christoffel symbols,

$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{3.4.76}$$

The divergence of the gauge field is,

$$\begin{aligned} \nabla^a n_a &= g^{ab}\partial_b n_a + \frac{1}{R^2}\cot\theta\, n_\theta \\ &= \frac{1}{R}\sum_{jm}\frac{1}{\sqrt{j(j+1)}}\Big\{t_{jm}\Big(\partial_\theta(-\csc\theta\,\partial_\phi Y_{jm}) - \cot\theta\csc\theta\,\partial_\phi Y_{jm} \\ &\quad + \csc\theta\,\partial_\phi\partial_\theta Y_{jm}\Big) + s_{jm}\Big(\partial_\theta\partial_\theta Y_{jm} + \cot\theta\,\partial_\theta Y_{jm} + \csc^2\theta\,\partial_\phi\partial_\phi Y_{jm}\Big)\Big\} \\ &= R\sum_{jm}\frac{1}{\sqrt{j(j+1)}}\, s_{jm}\Delta_{S^2} Y_{jm} \end{aligned} \tag{3.4.77}$$

By applying the gauge condition $\nabla_a n^a = 0$, the complex coefficient $s_{jm} = 0$. The orthonormality condition for the remaining vector harmonic $T_{lm\,a}$ is,

$$\int d\Omega\, T^\dagger_{lm\,a} T^a_{l'm'} = \delta_{ll'}\delta_{mm'} \tag{3.4.78}$$

It is an eigenfunction of the total angular momentum $\hat{L}^2$ and orbital angular momentum L^2,

$$L^2 T_{lm\,a} = l(l+1) T_{lm\,a} \tag{3.4.79}$$

It follows that any gauge field on the 2-sphere can be expanded in the vector harmonic $T_{lm\,a}$,

$$n_a(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} n_{lm} T_{lm\,a}(\theta, \phi) \tag{3.4.80}$$

3.5 Twisted Compactification

A problem is encountered when compactifying a supersymmetric gauge theory on a 2-sphere. The supersymmetry algebra (2.1.7a) states that bosonic and fermionic superpartners must have equal mass. In a gauge theory the gauge bosons are massless, so in order to have a supersymmetric gauge theory there must exist massless fermionic superpartners to the gauge bosons. The calculation in Section 3.4.3 found that the spectrum of the Dirac operator on the 2-sphere, equation (3.4.58), contains no massless mode. There are no massless fermions on the 2-sphere. The compactification of a supersymmetric gauge theory on a 2-sphere breaks all supersymmetries [14, 10, 11].

The reason why supersymmetry is broken on the 2-sphere can be observed directly. Consider a pure $U(1)$ $\mathcal{N}=1$ SUSY Yang-Mills theory on a flat manifold, with the Lagrangian [46],

$$\mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} i \bar{\Psi} \Gamma^a \partial_a \Psi \tag{3.5.1}$$

The supersymmetry transformations for these fields are [46, 27],

$$\delta A_a = \frac{i}{2} \left(\bar{\xi} \Gamma_a \Psi - \bar{\Psi} \Gamma_a \xi \right) \tag{3.5.2a}$$

$$\delta \Psi = M^{ab} F_{ab}\, \xi \tag{3.5.2b}$$

$$\delta \bar{\Psi} = -\bar{\xi}\, M^{ab} F_{ab} \tag{3.5.2c}$$

Under this supersymmetry transformation the Lagrangian is invariant up to a total derivative. Now consider this Lagrangian for a curved manifold. A vielbein, spin connection and covariant derivative are introduced. The Grassmann-valued parameters ξ

are now functions of spacetime $\xi(x)$ [46]. If the supersymmetry transformation above is repeated on a curved manifold then the variation of the Lagrangian must be zero, up to a total derivative, in order for supersymmetry to be preserved. Concentrating on only the terms dependent on the covariant derivatives of $\xi(x)$, then the variation in the Lagrangian is [46],

$$\delta\mathcal{L} = e^a_\alpha e^b_\beta e^c_\gamma \Big\{ (\nabla_a \bar{\xi}) M^{\beta\gamma} \Gamma^\alpha \Psi + \bar{\Psi} \Gamma^\alpha M^{\beta\gamma} (\nabla_a \xi) \Big\} F_{bc} \tag{3.5.3}$$

In order to preserved supersymmetry on a curved spacetime manifold the Grassmann-valued parameters must satisfy,

$$\nabla_a \xi(x) = \left(\partial_a + \frac{1}{2} R^{\alpha\beta}_a M_{\alpha\beta} \right) \xi(x) = 0 \tag{3.5.4}$$

This is the Killing equation for spinor fields. For supersymmetry to be preserved the curved manifold must admit Killing spinors (covariantly constant spinors). There are no non-trivial solutions to the Killing equation on the 2-sphere, hence supersymmetry is completely broken upon spherical compactification.

Some supersymmetries of the flat theory can be preserved upon compactification to a 2-sphere (or any curved manifold), by performing a topological twist [14, 10, 11]. The spin connection is embedded in the R-symmetry of the theory, via an external gauge field B_a which is coupled to the R-symmetry. Recall the geometry of gauge invariance [2]. In order to define a gauge covariant derivative, fields at points separated by infinitesimal distance must be related with a comparator $U(y, x)$ [2].

$$D_a \Psi(x) \delta x^a = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\Psi(x + \epsilon\, \delta x) - U(x + \epsilon\, \delta x, x) \Psi(x) \right] \tag{3.5.5}$$

The comparator quantifies the change in phase of a spinor when moved from a point x to $x + \delta x$.

$$\Psi(x + \delta x) = U(x + \delta x, x) \Psi(x) \tag{3.5.6}$$

The comparator on a path Γ is given by the Wilson line,

$$U(\Gamma) = \mathcal{P} \exp \left(- \int_\Gamma dx^a B_a(x) \right) \tag{3.5.7}$$

where $\mathcal{P}$ is a path-ordering for the integral. Therefore the gauge covariant derivative is [2],

$$D_a\Psi(x)\delta x^a = \partial_a\Psi(x)\delta x^a + B_a(x)\delta x^a \tag{3.5.8}$$

If the spin connection is embedded in the R-symmetry the Wilson line becomes,

$$U(\Gamma) = \mathcal{P}\exp\left(-\int_\Gamma dx^a\big(R_a(x) + B_a(x)\big)\right) \tag{3.5.9}$$

where $R_a = \frac{1}{2}\, R_a^{\alpha\beta} M_{\alpha\beta}$. The Killing equation for spinors is now,

$$\left(\partial_a + \frac{1}{2}\, R_a^{\alpha\beta} M_{\alpha\beta} + B_a\right)\xi(x) = 0 \tag{3.5.10}$$

By identifying $R_a = -B_a$ the Killing equation reduces to,

$$\partial_a\xi(x) = 0 \tag{3.5.11}$$

This is solved by a constant spinor. The 2-sphere admits the presence of constant spinors and therefore some of the supersymmetries are preserved.

From a field theory perspective this topological twisted appears artificial, however it does have a geometrical explanation from the D-brane perspective. The compactification of the gauge theory on the 2-sphere corresponds to partially wrapping a D-brane on a topologically non-trivial q-dimensional cycle of a non-compact Calabi-Yau manifold. The spin connection is the connection for the q-cycle. The R-symmetry of the theory decomposes into two parts, one part fills out the remaining non-trivial group structure of the Calabi-Yau manifold, the other is the trivial transverse flat part. The gauge field B_a is the connection of the R-symmetry on the Calabi-Yau manifold. The topological twist is performed in this description by taking a subgroup of the R-symmetry on the Calabi-Yau manifold $SO(X)_R$ and identifying it with the 'Lorentz' symmetry of the q-cycle $SO(q)$. In this Thesis, spherical compactification with a topological twist of the type studied by Maldacena and Núñez [14] is refered to as *twisted compactification.* The twisted compactification of a gauge theory on a 2-sphere will be demonstrated in the next Chapter.

Chapter 4

The Maldacena-Núñez Compactification

This Chapter will examine the Maldacena-Núñez compactified gauge theory dual to the Maldacena-Núñez background. The gauge theory is the low-energy ($\alpha' \to 0$) theory on the worldvolume of D5-branes or NS5-branes wrapped on a non-trival 2-cycle of a CY_3. The low-energy theory on the worldvolume of D5-branes is the six-dimensional $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory, therefore the Maldacena-Núñez compactified gauge theory will be constructed via the twisted compactification of the $\mathcal{N} = (1,1)$ theory. Section 4.1 presents the compactification from a group theory perspective, illustrating the decomposition and re-definition of the $\mathcal{N} = (1,1)$ fields. Section 4.2 will construct the (bosonic) action of the Maldacena-Núñez compactified gauge theory from the action of the $\mathcal{N} = (1,1)$ theory constructed in Section 3.2. Finally, Section 4.3 will calculate the Kaluza-Klein spectrum of the Maldacena-Núñez compactified gauge theory by integrating out the dimensions compactified on the 2-sphere.

4.1 Group Structure

The starting point of the Maldacena-Núñez compactified gauge theory is the $\mathcal{N} = (1,1)$ theory which is defined on a six-dimensional Minkowski spacetime $\Re^{5,1}$, via dimensional reduction. The $\mathcal{N} = (1,1)$ theory has a global group structure.

$$G = SO(5,1) \times SO(4) \sim SO(5,1) \times SU(2)_A \times SU(2)_B \tag{4.1.1}$$

In a six-dimensional Minkowski spacetime the Lorentz group is $SO(5,1)$ and the global group $SO(4)$ defines the R-symmetry of the $\mathcal{N} = (1,1)$ superalgebra. The field content of the $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory is a six-dimensional gauge field A_i (the longitudinal components of ten-dimensional gauge field), four real scalar fields ϕ_m (the transverse components of ten-dimensional gauge field) and four Weyl spinors, two of each opposing chirality $\lambda^A_{\underline{\alpha}}$, $\bar{\lambda}^{\underline{\dot{\alpha}}}_A$.[1] Under the global symmetries (4.1.1), the six-dimensional fields transform under the following representations.

	$SU(4)$	$SU(2)_A$	$SU(2)_B$
A_i	**6**	**1**	**1**
ϕ_m	**1**	**2**	**2**
$\lambda^A_{\underline{\alpha}}$	**4**	**2**	**1**
$\bar{\lambda}^{\underline{\dot{\alpha}}}_A$	$\bar{\mathbf{4}}$	**1**	**2**

The first step in compactifying the $\mathcal{N} = (1,1)$ theory is to decompose the six-dimensional spacetime manifold to a sub-manifold.

$$\Re^{5,1} \to \Re^{3,1} \times \Re^2 \tag{4.1.2}$$

The decomposition of the manifold induces a decomposition of the Lorentz group to a subgroup.

$$SO(5,1) \to H = SO(3,1) \times SO(2) \tag{4.1.3}$$

The subgroup H has the universal covering group,[2]

$$\tilde{H} = SU(2)_L \times SU(2)_R \times U(1)_{45} \tag{4.1.4}$$

[1]The spacetime index $i = 0, 1, \ldots, 5$; $m = 1, 2, 3, 4$; $SO(5,1)$ spinor index $A = 1, 2, 3, 4$ and $SU(2)_A \times SU(2)_B$ indices $\underline{\alpha}, \underline{\dot{\alpha}} = 1, 2$.

[2]A Lie algebra can have many associated Lie groups, there is not a 1-1 correspondence between Lie algebras and Lie groups. A Lie group with a simply connected domain is called the universal covering group [47].

Under the decomposition of the Lorentz group, the representations of the Lorentz group decompose into representations of the subgroup H. The six-dimensional gauge field A_i decomposes as,

$$\begin{aligned} SO(5,1) &\rightarrow SO(3,1)\times SO(2) \\ \mathbf{6} &\rightarrow (\mathbf{4},\mathbf{1})\oplus(\mathbf{1},\mathbf{2}) \end{aligned}$$

The six-dimensional gauge field decomposes into a four-dimensional vector field (under the four-dimensional Lorentz group $SO(3,1)$) and two scalar fields (which under the 'Lorentz' group of the 2-plane $SO(2)$, form a 2-vector).

$$A_i = \begin{cases} A_\mu & i=0,1,2,3=\mu \\ n_a & i=4,5=a+3 \end{cases} \tag{4.1.5}$$

for $a=1,2$. The 2-vector can be expressed as two complex scalar fields by the definition,

$$n_\pm = \frac{1}{\sqrt{2}}\,(n_1 \pm i\, n_2) \tag{4.1.6}$$

which expresses the 2-vector n_a in terms of its covering group $U(1)_{45}$.

The fields ϕ_m are real scalars under the Lorentz group $SO(5,1)$ and form a 4-vector under the R-symmetry $SO(4)$. $SO(4)$ is locally isomorphic to $SU(2)_A\times SU(2)_B$ and the 4-vector can be expressed as a bispinor.

$$\begin{aligned} SO(4) &\rightarrow SU(2)_A\times SU(2)_B \\ \mathbf{4} &\rightarrow (\mathbf{2},\mathbf{2}) \end{aligned}$$

The fields $\lambda^A_{\underline{\alpha}}$ and $\bar\lambda^{\dot{\underline{\alpha}}}_A$ are spinor representations of $SU(4)$, $\mathbf{4}$ and $\bar{\mathbf{4}}$ respectively. In six spacetime dimensions each spinor has 8-components and has the following decomposition under the covering group.

$$\begin{aligned} SU(4) &\rightarrow SU(2)_L\times SU(2)_R\times U(1)_{45} \\ \mathbf{4} &\rightarrow (\mathbf{2},\mathbf{1})^{+1}\oplus(\mathbf{1},\mathbf{2})^{-1} \\ \bar{\mathbf{4}} &\rightarrow (\mathbf{2},\mathbf{1})^{-1}\oplus(\mathbf{1},\mathbf{2})^{+1} \end{aligned}$$

where the superscript denotes $U(1)_{45}$ charge. The two left-handed Weyl spinors of $SO(5,1)$ $\lambda^A_{\underline{\alpha}}$ decompose into two left-handed Weyl spinors of $SO(3,1)$ and two right-handed Weyl spinors of $SO(3,1)$,

$$\lambda^A_{\underline{\alpha}} \to \lambda^\alpha_{\underline{\alpha}} \oplus \bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}} \tag{4.1.7}$$

Similarly, the two right-handed Weyl spinors of $SO(5,1)$ $\bar{\lambda}^{\dot{\underline{\alpha}}}_A$ decompose into two left-handed Weyl spinors of $SO(3,1)$ and two right-handed spinors of $SO(3,1)$,

$$\bar{\lambda}^{\dot{\underline{\alpha}}}_A \to \psi^{\alpha}_{\dot{\underline{\alpha}}} \oplus \bar{\psi}^{\dot{\alpha}}_{\dot{\underline{\alpha}}} \tag{4.1.8}$$

Note that indices α, $\dot{\alpha}$ are the usual $SU(2)_L \times SU(2)_R$ indices and $\underline{\alpha}$, $\dot{\underline{\alpha}}$ are the $SU(2)_A \times SU(2)_B$ indices. Summarising these decompositions, the bosons transform under the following representations of the subgroup H.

	$SU(2)_L$	$SU(2)_R$	$U(1)_{45}$	$SU(2)_A$	$SU(2)_B$
A_μ	**2**	**2**	0	**1**	**1**
$n_\pm$	**1**	**1**	± 2	**1**	**1**
ϕ_m	**1**	**1**	0	**2**	**2**

The fermions transform under the representations,

	$SU(2)_L$	$SU(2)_R$	$U(1)_{45}$	$SU(2)_A$	$SU(2)_B$
$\lambda^\alpha_{\underline{\alpha}}$	**2**	**1**	$+1$	**2**	**1**
$\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}}$	**1**	**2**	-1	**2**	**1**
$\psi^\alpha_{\dot{\underline{\alpha}}}$	**2**	**1**	-1	**1**	**2**
$\bar{\psi}^{\dot{\alpha}}_{\dot{\underline{\alpha}}}$	**1**	**2**	$+1$	**1**	**2**

The 2-plane must be compactified to a 2-sphere. As discussed in Section 3.5, a conventional compactification on the 2-sphere breaks all supersymmetries. To preserve some supersymmetry a topological twist is performed upon compactification. In a conventional compactification the 'spin' of the fields on the 2-sphere is given by the local 'Lorentz' group, $SO(2) \sim U(1)_{45}$. In the twisted compactification, the group $U(1)_{45}$ is embedded in a non-trivial subgroup of the R-symmetry. There are two inequivalent embeddings of the spin-connection in the R-symmetry [10]. The group $U(1)_{45}$ can be embedded into the diagonal subgroup of the R-symmetry.

$$U(1)_T = D\Big(U(1)_{45} \times U(1)_D\Big)$$

where $U(1)_D = D\big(SU(2)_A \times SU(2)_B\big)$ and the D denotes the diagonal subgroup. This embedding preserves one-half of the supersymmetries (i.e. eight) giving a four-dimensional field theory with $\mathcal{N} = 2$ supersymmetry [10, 48]. Embedding the spin connection in the subgroup $U(1)_A \subset SU(2)_A$,

$$U(1)_T = D\Big(U(1)_{45} \times U(1)_A\Big)$$

preserves a quarter of the supersymmetries (i.e. four) and leads to a four-dimensional field theory with $\mathcal{N} = 1$ supersymmetry [14, 10].[3] This is the Maldacena-Núñez compactified gauge theory.

With the embedding defined the representation of each field under the group $U(1)_T$ can be determined. Each $U(1)$ group has an associated generator Q, hence $Q_T = Q_{45} + Q_A$, where the values $Q_A = \pm 1$ have been normalised for states in the fundamental representation of $SU(2)_A$. The fields transform under the following representations of $U(1)_T$.

	$U(1)_{45}$	$U(1)_A$	$U(1)_T$
A_μ	0	0	0
$n_\pm$	± 2	0	± 2
ϕ_i	0	± 1	± 1
$\lambda^{\alpha}_{\underline{\alpha}}$	$+1$	± 1	$\begin{pmatrix} +2 \\ 0 \end{pmatrix}$
$\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}}$	-1	± 1	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$
$\psi^{\alpha}_{\underline{\dot{\alpha}}}$	-1	0	-1
$\bar{\psi}^{\dot{\alpha}}_{\underline{\dot{\alpha}}}$	$+1$	0	$+1$

[3] A physically identical theory is obtained by the embedding the spin connection in $U(1)_T = D\big(U(1)_{45} \times U(1)_B\big)$.

The twisted compactification of the 2-plane to the 2-sphere is performed by assigning $U(1)_T$ as the local 'Lorentz' group and refering to the $U(1)_T$ quantum numbers as T-spin. The fields under $U(1)_T$ are,

T-scalars:	$Q_T = 0$	A_μ, $\lambda^{\alpha}_{\underline{\alpha}=2}$, $\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}=1}$
T-spinors:	$Q_T = \pm 1$	$\psi^{\alpha}_{\underline{\dot{\alpha}}}$, $\bar{\psi}^{\dot{\alpha}}_{\underline{\dot{\alpha}}}$, ϕ_i
T-vectors:	$Q_T = \pm 2$	$n_\pm$, $\lambda^{\alpha}_{\underline{\alpha}=1}$, $\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}=2}$

The usual terms scalar, spinor and vector refer to the transformation properties of the fields under the four-dimensional Lorentz group $SO(3,1)$. Under the twisted compactification of Maldacena and Núñez there exists a fermionic T-scalar which is also a four-dimensional Weyl spinor. The presence of this Weyl spinor allows the existance of a fermion with a zero eigenvalue on the 2-sphere to preserve four supersymmetries.

4.2 The Bosonic Action

In this Section the twisted compactification of Maldacena and Núñez will be applied to the (bosonic) action of the $\mathcal{N} = (1,1)$ theory to construct the (bosonic) action of the compactified gauge theory. This action will subsequently be compared with the effective six-dimensional action of the Higgsed $\mathcal{N} = 1^*$ theory presented in Section 6.2.

The bosonic action of the $U(1)$ $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory was calculated in Section 3.2.

$$\mathcal{S}_B = \frac{1}{g_6^2} \int d^6x \left\{ -\frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} \partial_i \phi_n \partial^i \phi^n \right\} \tag{4.2.1}$$

The indices $i, j = 0, 1, \ldots, 5$ label the six spacetime dimensions and $m, n = 1, \ldots, 4$ label the R-symmetry dimensions. The decomposition of the spacetime manifold

(4.1.2) induces a decomposition of the bosonic action $\mathcal{S}_B$.

$$
\begin{aligned}
\mathcal{S}_B = & \frac{1}{g_6^2}\int d^6x \Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{\mu a}F^{\mu a} - \frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}\partial_\mu\phi_m\partial^\mu\phi^m \\
& - \frac{1}{2}\partial_a\phi_m\partial^a\phi^m \Big\} \\
= & \frac{1}{g_6^2}\int d^6x \Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu n_a\partial^\mu n^a - \partial_\mu n_a\partial^a A^\mu - \frac{1}{2}\partial_a A_\mu\partial^a A^\mu \\
& - \frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}\partial_\mu\phi_m\partial^\mu\phi^m - \frac{1}{2}\partial_a\phi_m\partial^a\phi^m \Big\}
\end{aligned}
$$

In performing the twisted compactification to a 2-sphere, the group $U(1)_T \sim SO(2)_T$ acts as the local rotation group. The group structure in Section 4.1 states that A_μ is a T-scalar, the ϕ_m form T-spinors and the n_a form a T-vector. In moving from a flat spacetime to a curved spacetime, derivatives on the flat spacetime become general covariant derivatives. The derivatives corresponding to the 2-plane are therefore transformed into general covariant derivatives on the 2-sphere, $\partial_a \to \nabla_a$, whilst the derivatives corresponding to the flat four dimensions are unchanged $\partial_\mu \to \partial_\mu$.

$$F_{\mu\nu} \quad \to \quad F_{\mu\nu} \tag{4.2.2a}$$

$$\partial_\mu n_a \quad \to \quad \partial_\mu n_a \tag{4.2.2b}$$

A general covariant derivative's action on a scalar is that of an ordinary derivative.

$$\partial_a A_\mu \quad \to \quad \nabla_a A_\mu = \partial_a A_\mu \tag{4.2.3}$$

The action of a general covariant derivative on a vector is,

$$\partial_a n_b \to \nabla_a n_b = \partial_a n_b - \Gamma^c_{ab} n_c \tag{4.2.4}$$

where Γ^c_{ab} is a Christoffel symbol (3.3.13).

$$
\begin{aligned}
F_{ab} = \partial_a n_b - \partial_b n_a \quad \to \quad \mathcal{F}_{ab} &= \partial_a n_b - \Gamma^c_{ab} n_c - \partial_b n_a + \Gamma^c_{ba} n_c \\
&= \partial_a n_b - \partial_b n_a
\end{aligned}
\tag{4.2.5}
$$

The rank 2 tensor $\mathcal{F}_{ab}$ is the field tensor for the vector field n_a on the 2-sphere.

Consider the real scalars ϕ_m of $SO(4)$. The group structure of the Maldacena-Núñez compactified gauge theory indicates that these scalars form T-spinors on the 2-sphere. This identification stems from their transformation properties under $U(1)_A \subset SU(2)_A$. The transformation properties of the scalars under $SU(2)_A \times SU(2)_B \sim SO(4)$ are explicitly revealed by the construction of a $SO(4)$ bispinor, see Appendix A.

$$\begin{aligned} v_{\underline{\alpha}\dot{\alpha}} &= i(\tau^m)_{\underline{\alpha}\dot{\alpha}}\phi_m \\ \phi^m &= -\frac{i}{2}\,(\bar{\tau}^m)^{\dot{\alpha}\underline{\alpha}}\,v_{\underline{\alpha}\dot{\alpha}} \end{aligned} \tag{4.2.6}$$

Substituting this expression for the real scalars into the last term of the bosonic action,

$$\begin{aligned} \int d^2x\,\partial_a\phi_m\partial^a\phi^m &= -\frac{1}{4}\int d^2x\,\partial_a\Big((\bar{\tau}^m)^{\dot{\alpha}\underline{\alpha}}\,v_{\underline{\alpha}\dot{\alpha}}\Big)\partial^a\Big((\bar{\tau}_m)^{\dot{\beta}\underline{\beta}}\,v_{\underline{\beta}\dot{\beta}}\Big) \\ &= \frac{1}{2}\int d^2x\,\partial_a v^{\underline{\alpha}\dot{\alpha}}\,\partial^a v_{\underline{\alpha}\dot{\alpha}} \\ &= \frac{1}{2}\int d^2x\,v^{\underline{\alpha}}{}_{\dot{\alpha}}\,\partial_a\partial^a v_{\underline{\alpha}}{}^{\dot{\alpha}} \end{aligned}$$

If a 2-component object $\Xi^{\hat{\alpha}}$ is defined from the $SU(2)_A$ components of the bispinor,

$$\Xi^{\hat{\alpha}} = \begin{pmatrix} v_{\underline{1}}{}^{\dot{\alpha}} \\ v_{\underline{2}}{}^{\dot{\alpha}} \end{pmatrix} \tag{4.2.7}$$

then,

$$\frac{1}{2}\int d^2x\,v^{\underline{\alpha}}{}_{\dot{\alpha}}\,\partial_a\partial^a v_{\underline{\alpha}}{}^{\dot{\alpha}} = -\frac{1}{2}\int d^2x\,\Xi^\dagger_{\hat{\alpha}}\delta^{\hat{\alpha}}{}_{\hat{\beta}}\partial_a\partial^a\,\Xi^{\hat{\beta}} \tag{4.2.8}$$

as $(v_{\underline{\alpha}}{}^{\dot{\alpha}})^\dagger = (\lambda^A_{\underline{\alpha}}\bar{\lambda}^{\dot{\alpha}}_A)^\dagger = -v^{\underline{\alpha}}{}_{\dot{\alpha}}$. Note that $\hat{\alpha}$ labels the two components of Ξ. The non-trivial $U(1)_A$ structure has been revealed by the definition of $\Xi^{\hat{\alpha}}$, as the term (4.2.8) is invariant under the global transformation,

$$\Xi^{\hat{\alpha}} \rightarrow e^{i\gamma}\Xi^{\hat{\alpha}} \qquad \Xi^\dagger_{\hat{\alpha}} \rightarrow e^{-i\gamma}\Xi^\dagger_{\hat{\alpha}}$$

for a constant parameter γ. The differential operator can be re-written as the square of the Dirac operator on the 2-plane.

$$\delta^{\hat{\alpha}}{}_{\hat{\beta}}\,\partial_a\partial^a = \begin{pmatrix} \partial_a\partial^a & 0 \\ 0 & \partial_a\partial^a \end{pmatrix}^{\hat{\alpha}}{}_{\hat{\beta}} = (\sigma^a)^{\hat{\alpha}}{}_{\hat{\gamma}}\partial_a(\sigma^b)^{\hat{\gamma}}{}_{\hat{\beta}}\partial_b = (\partial\!\!\!/^2)^{\hat{\alpha}}{}_{\hat{\beta}} \tag{4.2.9}$$

Upon the compactification of the 2-plane to the 2-sphere, the Dirac operator on the 2-plane is replaced with the Dirac operator on the 2-sphere $\partial\!\!\!/ \to \hat{\nabla}_{S^2}$.

$$\begin{aligned}-\frac{1}{2}\int d^2x\, \Xi^\dagger_{\hat{\alpha}}(\partial\!\!\!/^2)^{\hat{\alpha}}{}_{\hat{\beta}}\Xi^{\hat{\beta}} \to -\frac{1}{2}\int d\Omega\, \Xi^\dagger_{\hat{\alpha}}(\hat{\nabla}^2_{S^2})^{\hat{\alpha}}{}_{\hat{\beta}}\Xi^{\hat{\beta}} \\ = \frac{1}{2}\int d\Omega\, \Xi^\dagger_{\hat{\alpha}}[(-i\hat{\nabla}_{S^2})^2]^{\hat{\alpha}}{}_{\hat{\beta}}\Xi^{\hat{\beta}}\end{aligned} \tag{4.2.10}$$

The kinetic term for complex scalars is derived through a similar calculation.

$$\partial_\mu\phi_m\partial^\mu\phi^m = \frac{1}{2}\partial_\mu\Xi^\dagger_{\hat{\alpha}}\partial^\mu\Xi^{\hat{\alpha}} \tag{4.2.11}$$

In summary, the bosonic action of the Maldacena-Núñez compactification is,

$$\begin{aligned}\mathcal{S}_B =& \frac{1}{g_6^2}\int d^4x\int R^2 d\Omega\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu n_a\partial^\mu n^a - \frac{1}{2}\partial_a A_\mu\partial^a A^\mu \\ &- \frac{1}{4}\mathcal{F}_{ab}\mathcal{F}^{ab} - \frac{1}{4}\partial_\mu\Xi^\dagger_{\hat{\alpha}}\partial^\mu\Xi^{\hat{\alpha}} - \frac{1}{4}\Xi^\dagger_{\hat{\alpha}}[(-i\hat{\nabla}_{S^2})^2]^{\hat{\alpha}}{}_{\hat{\beta}}\Xi^{\hat{\beta}}\Big\}\end{aligned} \tag{4.2.12}$$

with the 'Lorentz' gauge $\nabla_a n^a$ imposed.

4.3 Classical Kaluza-Klein Spectrum

Finally in this Section the classical Kaluza-Klein spectrum of the Maldacena-Núñez compactified gauge theory will be calculated. In principle, the Kaluza-Klein spectrum is determined by integrating out the compact dimensions to obtain an effective four-dimensional theory. The compact dimensions form the mass terms of the effective four-dimensional action. The Kaluza-Klein spectrum of each field with a specific T-spin can be calculated by evaluating the kinetic term of one such field on the 2-sphere.

The T-scalars comprise of a gauge field A_μ, a left-handed Weyl spinor $\lambda^\alpha_{\underline{\alpha}=2}$ and a right-handed Weyl spinor $\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}=1}$, in four spacetime dimensions. A scalar field on the 2-sphere has the action (3.4.28), therefore the action of the massless real scalar field A_μ on a 2-sphere of radius R is,

$$\mathcal{S}_A = \frac{1}{g_6^2}\int R^2 d\Omega\, A_\mu\Delta_{S^2}A^\mu \tag{4.3.1}$$

A scalar field on the 2-sphere can be expanded in terms of spherical harmonics,

$$A_\mu(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{(\mu)lm} Y_{lm}(\theta,\phi) \tag{4.3.2}$$

In Section 3.4.2 the eigenvalues of the scalar Laplacian were shown to be $\sim -l(l+1)$, for integer $l \geq 0$ and $m = -l, \ldots, l$, with a degeneracy $2l+1$. Under the expansion of the scalar field in spherical harmonics,

$$\begin{aligned} \mathcal{S}_A &= \frac{1}{g_6^2}\int R^2 d\Omega \sum_{l,m,l',m'} \bar{A}_{(\mu)lm} A^{(\mu)}_{l'm'}\, Y^\dagger_{lm}\Delta_{S^2} Y_{l'm'} \\ &= -\frac{1}{g_6^2}\sum_{l,m,l',m'} \bar{A}_{(\mu)lm} A^{(\mu)}_{l'm'}\, l(l+1)\,\delta_{ll'}\delta_{mm'} \end{aligned} \tag{4.3.3}$$

This shows that the gauge boson A_μ has a Kaluza-Klein tower of states with mass (squared),

$$M^2 = \frac{1}{R^2}\, l(l+1) \tag{4.3.4}$$

and degeneracy $2l+1$. The supersymmetric partner to the gauge boson comprises of a left-handed Weyl spinor and a right-handed Weyl spinor. By supersymmetry these Weyl spinors have the same Kaluza-Klein tower of states as the gauge boson.

The T-spinor fields comprise of two left-handed Weyl spinors $\psi^\alpha_{\underline{\alpha}}$, two right-handed Weyl spinors $\bar{\psi}^{\dot{\alpha}}_{\underline{\alpha}}$ and four real scalars ϕ_m. The action of a two-component Dirac spinor Υ on the 2-sphere is known to have the action (3.4.49). The Dirac spinor on the 2-sphere is expanded in the spherical spinors (of the spherical basis),

$$\Upsilon = \sum_{jm}\sum_{+,-} v^\pm_{jm}\Upsilon^\pm_{jm} \tag{4.3.5}$$

The action of the fermionic T-spinor on the 2-sphere is,

$$\begin{aligned} \mathcal{S}_\Upsilon &= \frac{1}{g_6^2}\int R^2 d\Omega\, \bar{\Upsilon}\,(-i\nabla_{S^2})\,\Upsilon \\ &= \sum_{jm,j'm'}\sum_{\epsilon,\epsilon'} \bar{v}^\epsilon_{jm} v^{\epsilon'}_{j'm'} \int R^2 d\Omega\, \left(\Upsilon^\epsilon_{jm}\right)^\dagger (-i\nabla_{S^2})\,\Upsilon^{\epsilon'}_{j'm'} \\ &= \sum_{jm,j'm'}\sum_{\epsilon,\epsilon'} \bar{v}^\epsilon_{jm} v^{\epsilon'}_{j'm'}\, \epsilon\left(j+\frac{1}{2}\right)\delta_{jj'}\delta_{mm'}\delta^{\epsilon\epsilon'} \end{aligned} \tag{4.3.6}$$

The total angular momentum quantum number $j = \frac{1}{2}, \frac{3}{2}, \ldots$, with degeneracy $2j+1$. By defining the quantum number $l = j + \frac{1}{2}$ (of orbital angular momentum [38]),

$$\mathcal{S}_\Upsilon = \sum_{lm,l'm'} \sum_{\epsilon,\epsilon'} \bar{v}^{\epsilon}_{jm} v^{\epsilon'}_{j'm'}\, \epsilon\, l\, \delta_{ll'} \delta_{mm'} \delta^{\epsilon\epsilon'} \tag{4.3.7}$$

Each Dirac spinor Υ has a Kaluza-Klein tower of states with mass (squared),

$$M^2 = \frac{1}{R^2}\, l^2 \tag{4.3.8}$$

with integer $l \geq 1$ and degeneracy $2l$. For each mass M^2, there are $4l$ left-handed and $4l$ right-handed Weyl spinors of $SO(3,1)$. Supersymmetry implies that the bosonic T-spinors ϕ_m have the same Kaluza-Klein tower of states.

Finally, the T-vectors comprise of two real scalars n_a which form a 2-vector, a left-handed Weyl spinor and a right-handed Weyl spinor. The action of a 2-vector field n_a on the 2-sphere is given in equation (3.4.73).

$$\mathcal{S}_n = -\frac{1}{4g_6^2} \int R^2 d\Omega\, \mathcal{F}_{ab} \mathcal{F}^{ab} = -\frac{1}{2g_6^2} \int d\Omega\, \frac{1}{R^2} \csc^2\theta\, \mathcal{F}_{\theta\phi} \mathcal{F}_{\theta\phi} \tag{4.3.9}$$

where $\mathcal{F}_{ab} = \partial_a n_b - \partial_b n_a$. The (gauge-fixed) T-vector can be expanded in the vector harmonics,

$$n_a = \sum_{lm} n_{lm} T_{lm\,a} \tag{4.3.10}$$

Under this expansion the field tensor becomes,

$$\begin{aligned} \mathcal{F}_{\theta\phi} &= R \sum_{l,m} n_{lm} \frac{1}{\sqrt{l(l+1)}} \Big(\partial_\theta (\sin\theta\, \partial_\theta Y_{lm}) + \csc\theta\, \partial_\phi \partial_\phi Y_{lm} \Big) \\ &= -R \sum_{l,m} n_{lm} \frac{1}{\sqrt{l(l+1)}} \sin\theta\, L^2 Y_{lm} \end{aligned} \tag{4.3.11}$$

The action becomes,

$$\begin{aligned} \mathcal{S}_n &= -\frac{1}{2g_6^2} \sum_{l,m,l',m'} n^\dagger_{lm} n_{l'm'} \int d\Omega \frac{1}{\sqrt{l(l+1) l'(l'+1)}}\, Y^\dagger_{lm} L^4 Y_{l'm'} \\ &= -\frac{1}{2g_6^2} \sum_{l,m,l',m'} n^\dagger_{lm} n_{l'm'}\, l(l+1)\, \delta_{ll'} \delta_{mm'} \end{aligned} \tag{4.3.12}$$

The orbital angular momentum quantum number $l \geq 1$, with degeneracy $2l+1$. The T-vector n_a has a Kaluza-Klein tower of states with mass (squared),

$$M^2 = \frac{1}{R^2} l(l+1) \tag{4.3.13}$$

for integer $l \geq 1$ and degeneracy $2l+1$. By supersymmetry the left-handed Weyl fermion $\lambda^{\alpha}_{\underline{\alpha}=1}$ and right-handed Weyl fermion $\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}=2}$ have the same Kaluza-Klein tower of states.

The Kaluza-Klein spectrum has been determined for each particle of definite T-spin. In each case there is an infinite tower of states, each parameterised by an integer l. The Maldacena-Núñez compactification has four supercharges and hence has $\mathcal{N}=1$ supersymmetry in four dimensions. The particle content of the Maldacena-Núñez compactified gauge theory must fill multiplets of $\mathcal{N}=1$ supersymmetry. For $M^2=0$, there are only T-scalars that comprise of a four-dimensional vector field and a four-dimensional spinor,

$$(\mathbf{2},\mathbf{2}) \oplus \Big((\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\Big)$$

expressed in terms of $SU(2)_L \times SU(2)_R$ representations. The Maldacena-Núñez compactified gauge theory that was constructed had a $U(1)$ gauge group, therefore it must possess a $U(1)$ massless vector multiplet of $\mathcal{N}=1$ supersymmetry. This is exactly the particle content of the $M^2=0$ state. The massive states are summarised in the Table below.

T-spin	λ	States	Degeneracy
T-scalar	$l(l+1)$	$(\mathbf{2},\mathbf{2}) \oplus \Big((\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\Big)$	$(2l+1)$
T-spinor	l^2	$\Big((\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\Big) \oplus 2 \times (\mathbf{1},\mathbf{1})$	$4l$
T-vector	$l(l+1)$	$\Big((\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\Big) \oplus (\mathbf{1},\mathbf{1})$	$(2l+1)$

Quantum number $l \geq 1$. The remaining massive T-scalar states contain a vector field, therefore they must lie in a massive vector multiplet. However, the T-scalars do not have the particle content of a massive vector multiplet. The Table shows that both the T-scalars and the T-vectors have mass $M^2 \sim l(l+1)$, and together have the particle content of a massive vector multiplet of degeneracy $2l + 1$. Finally, the T-spinors have the particle content of a massive chiral multiplet of mass $M^2 \sim l^2$, with degeneracy $4l$. In summary, the theory contains the following $\mathcal{N} = 1$ multiplets.

Multiplet	Degeneracy
Massless Vector	1
Massive Vector	$2l + 1$
Massive Chiral	$4l$

The classical spectrum of a theory originates from terms that are quadratic in the fields. These terms are identical for a free theory and an interacting non-abelian gauge theory with adjoint matter. The classical spectrum of a free gauge theory will be identical to the spectrum of its non-abelian counterpart, except that the non-abelian theory will have a greater degeneracy for each mass M^2. The classical Kaluza-Klein spectrum of the Maldacena-Núñez compactified gauge theory with $U(p)$ gauge group is therefore,

Multiplet	Degeneracy
Massless Vector	p^2
Massive Vector	$(2l + 1)\, p^2$
Massive Chiral	$4l\, p^2$

For a $SU(p)$ gauge group, the extra degeneracy p^2 is replaced with $p^2 - 1$.

Chapter 5

Deconstruction

The theory constructed by Maldacena and Núñez is a big step towards the construction of a gravity dual of the four-dimensional $\mathcal{N} = 1$ SUSY Yang-Mills theory. In the IR, the Maldacena-Núñez background is dual to a $\mathcal{N} = 1$ SUSY Yang-Mills theory, but in the UV it is dual to the six-dimensional SUSY gauge theory constructed in Chapter 4. The Kaluza-Klein modes of the associated 2-sphere are at the same energy scale as $\Lambda_{\mathcal{N}=1}$, the scale at which the dynamics becomes strongly coupled. This makes it difficult to differentiate between the strong coupling dynamics (such as the mass of a glueball) and the Kaluza-Klein dynamics (such as the Kaluza-Klein modes of the glueball). It is not possible to decouple the Kaluza-Klein modes within the supergravity approximation [16]. Deconstruction offers an approach to identify the four-dimensional $\mathcal{N} = 1$ SUSY Yang-Mills theory dual to the full string solution of the Maldacena-Núñez background.

Deconstruction views the Kaluza-Klein modes of a higher-dimensional gauge theory as the massive states of a four-dimensional, spontaneously broken non-abelian gauge theory. In deconstruction, the Higgs phase of a four-dimensional gauge theory can be viewed as a theory with additional, discretised dimensions [17, 18]. The discretised nature of the extra dimensions provides the deconstructed theory with a natural UV cut-off. In the limit where the lattice spacing is reduced to zero the extra dimensions become continuous and the full higher-dimensional Lorentz invariance is restored. Deconstruction allows the non-renormalisable higher-dimensional gauge theories to be treated as a limit of a renormalisable four-dimensional gauge theory [17, 18]. Initially, deconstruction was demonstrated within the context of quiver gauge theories [17], theories with a composite gauge group containing adjoint and fundamental particles transforming under definite representations of the various component gauge groups. In string theory, quiver gauge theories are the low-energy theory on the worldvolume of D-branes probing orbifolds. A Dp-brane probing an orbifold deconstructs a higher-dimensional Dq-brane, $q > p$ [19]. Further work by Adams and Fabinger showed that D-branes probing orbifolds with discrete torsion deconstruct higher-dimensional D-branes with a non-commutative worldvolume [19]. D-branes with non-commutative worldvolumes can also be constructed in M(atrix) theory. Adams and Fabinger showed that these two approaches are equivalent.

The emergence of additional dimensions in deconstruction is clearly defined in the M(atrix) theory approach. M(atrix) theory takes a theory of $N \times N$ matrices, whose vacuum describes a matrix version of a spacetime (such as a torus or sphere). From a correspondence between matrices and functions this "fuzzy space" can be mapped to a non-commutative space. The expansion of the matrices about the vacuum constructs a field theory on the non-commutative space. The work by Dorey [20] was the first application of the matrix theory approach to deconstruction.[1] Dorey deconstructed a toroidally compactified LST using the β-deformed $\mathcal{N} = 4$ SUSY Yang-Mills theory. The starting point for deconstruction is the four-dimensional gauge theory. The Higgs vacuum of the β-deformed theory breaks the gauge group $U(N) \to U(p)$ and forms a fuzzy torus, a discrete non-commutative version of a 2-torus. M(atrix) theory identifies a correspondence between matrices on a fuzzy torus and functions on a non-commutative torus. By expanding the β-deformed theory about the vacuum and identifying the correspondence between matrices and functions, the β-deformed theory (in the limit $N \to \infty$) is equivalent to the toroidally compactified $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory [20]. Subsequent calculations show that in fact the β-deformed theory deconstructs the toroidally compactified LST (whose low-energy limit is the toroidally compactified $\mathcal{N} = (1,1)$ theory) [20].

Section 5.1 will demonstrate the construction of an additional two dimensions

[1] Besides the orginal paper by Adams and Fabinger [19].

in the $\mathcal{N} = 1^*$ SUSY Yang-Mills theory using M(atrix) theory techniques. The deconstruction technique will be applied explicitly in Chapter 6 to the Higgsed $\mathcal{N} = 1^*$ theory. In preparation for that Chapter, Section 5.2 will construct the Lagrangian of the Higgsed $\mathcal{N} = 1^*$ by applying the choice of Higgs vacuum to equation (2.6.5).

5.1 Extra Dimensions from M(atrix) Theory

This Section describes the appearance of extra dimensions in the $\mathcal{N} = 1^*$ SUSY Yang-Mills with $U(N)$ (or $SU(N)$) gauge group. The $\mathcal{N} = 1^*$ theory is a relevant deformation of the $\mathcal{N} = 4$ SUSY Yang-Mills theory. The superpotential of the $\mathcal{N} = 4$ theory is deformed by adding mass terms for the chiral multiplets giving the $\mathcal{N} = 1^*$ superpotential [22],

$$\mathcal{W}(\Phi) = g_{ym}\mathrm{Tr}\left(i\sqrt{2}\,\Phi_1[\Phi_2, \Phi_3] + \frac{\eta}{g_{ym}}\sum_{i=1}^{3}\Phi_i^2\right) \tag{2.6.1}$$

The theory has no moduli space, instead it contains a number of isolated vacua [21, 22]. The F-flatness condition for the $\mathcal{N} = 1^*$ theory is,

$$\frac{\partial \mathcal{W}}{\partial \Phi_i} = \frac{1}{\sqrt{2}}\, i\varepsilon_{ijk}[\Phi_j, \Phi_k] + 2\frac{\eta}{g_{ym}}\,\Phi_i = 0 \tag{5.1.1}$$

This gives the following relation between the complex scalar fields.

$$[\Phi_i, \Phi_j] = \sqrt{2}\frac{\eta}{g_{ym}}\, i\varepsilon_{ijk}\Phi_k \tag{5.1.2}$$

Under the reparameterisation,

$$\Phi_i \to \frac{g_{ym}}{\sqrt{2}\eta}\,\Phi_i \tag{5.1.3}$$

the F-flatness condition (5.1.2) becomes,

$$[\Phi_i, \Phi_j] = i\varepsilon_{ijk}\Phi_k \tag{5.1.4}$$

which is precisely the $SU(2)$ Lie algebra. The vacua must also solve the D-flatness condition,

$$\mathrm{Tr}[\Phi_i, \Phi_i^\dagger]^2 = 0 \tag{5.1.5}$$

The F-flatness and D-flatness conditions can be solved by any d-dimensional representation of the $SU(2)$ generators, which in general will be reducible. Representations of the gauge group $U(N) \sim SU(N) \times U(1)$ are $N \times N$ matrices.[2] There is a single irreducible representation $J_i^{(d)}$ of the $SU(2)$ Lie algebra for every dimension d, which allows the gauge group to be decomposed into a number of irreducible representations, of total dimension N [22]. If the number of times a representation d appears is denoted k_d, then the unbroken gauge group is $U(N) \to \otimes_d U(k_d)$ (or $SU(N) \to [\otimes_d U(k_d)]/U(1)$). A general Higgs branch has the vacuum $\Phi_i = \mathbb{1}_p \otimes J_i^{(q)}$, p copies of the q-dimensional representation of the $SU(2)$ Lie algebra, which breaks the gauge group $U(N = pq) \to U(p)$ (or $SU(N) \to SU(p)$) [22]. The Higgs vacuum is given by the special case of $q = N$, $p = 1$, where the gauge group is broken $U(N) \to U(1)$.

Extra dimensions emerge via the mechanism seen in M(atrix) theory. It is found that the Higgs vacuum describes a fuzzy sphere [24], a discretised version of the 2-sphere. In Section 3.4.1 a 2-sphere of radius R was constructed by embedding the manifold in $\Re^3$,

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{3.4.1}$$

Any function on the 2-sphere can be expanded in terms of the coordinates x_i by a Taylor expansion,

$$f(x) = f_0 + f_i x^i + \frac{1}{2} f_{ij}\, x^i x^j + \dots \tag{5.1.6}$$

The definition of the fuzzy (2-)sphere begins by truncating this expansion to the Nth term [24]. The truncation replaces the algebra $\mathcal{C}(S^2)$ of complex functions with a vector space $\mathcal{A}_N$. The dimension of the vector space $\mathcal{A}_N$ is dependent on the number of independent components in the truncated expansion. Let the number of components of a completely symmetric tensor $f_{a_1 \dots a_l}$ of rank l be denoted by N_l [24]. Equation (3.4.1) constrains the number of independent components at each order in the truncated expansion,

$$f(x) = f_0 + f_i x^i + \frac{1}{2} f_{ij}\, x^i x^j + \dots + \frac{1}{l!} f_{a_1 a_2 \dots a_l} x^{a_1} x^{a_2} \dots x^{a_l} \tag{5.1.7}$$

[2]For the gauge group $SU(N)$ the representations are $N \times N$ traceless matrices.

for $l \geq 2$, reducing the number of independent components at each order by N_{l-2}. Therefore the total number of independent components in a tensor of rank l, at each order in the truncated expansion (5.1.7) is,

$$N_l - N_{l-2} = 2l + 1$$

The dimension of the vector space $\mathcal{A}_N$ is the sum of independent components at all orders,

$$\sum_{l=0}^{N-1}(2l+1) = N^2$$

The vector space $\mathcal{A}_N$ can be replaced with an algebra $\mathcal{M}_N$ of complex $N \times N$ matrices by making the replacement [24],

$$x_i \longrightarrow \hat{x}_i = \tau J_i^{(N)} \tag{5.1.8}$$

where $\tau^2 = \frac{4R^2}{N^2-1}$ (and denoting all matrix fields with hats). From the $SU(2)$ Lie algebra the coordinates of the fuzzy sphere $\hat{x}_i$ have the commutation relation,

$$[\hat{x}_i, \hat{x}_j] = i\tau\, \varepsilon_{ijk}\hat{x}_k \tag{5.1.9}$$

In the limit $N \rightarrow \infty$ ($\tau \rightarrow 0$) the commutative 2-sphere is recovered. Equivalently, by making the replacement (5.1.8), equation (3.4.1) becomes,

$$\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = \mathbb{1} \tag{5.1.10}$$

the defining equation of the fuzzy sphere. From the definition of the fuzzy sphere, the Higgs vacuum of the $\mathcal{N} = 1^*$ theory can be expressed in terms of the coordinate matrices of the fuzzy sphere.

$$\Phi_i = J_i^{(N)} = \frac{1}{\tau}\,\hat{x}_i \tag{5.1.11}$$

The Higgs vacuum describes a fuzzy sphere defined by $N \times N$ coordinate matrices.[3]

The Higgsed $\mathcal{N} = 1^*$ theory is a theory of $N \times N$ matrices. In M(atrix) theory there is a correspondence between such matrix theories and non-commutative field

[3] In the more general Higgs vacua, the fuzzy sphere is defined by $q \times q$ coordinate matrices.

theories [23, 49]. Scalar functions on a 2-sphere can be expanded in terms of spherical harmonics,

$$a(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi) \tag{3.4.21}$$

The spherical harmonics can be expressed in terms of the cartesian coordinates x_A with $A = 1, 2, 3$ of a unit vector in $\Re^3$ [23, 49],

$$Y_{lm}(\theta,\phi) = R^{-l}\sum_{\vec{A}} f^{(lm)}_{A_1\ldots A_l}x^{A_1}\ldots x^{A_l} \tag{5.1.12}$$

where $f^{(lm)}_{A_1\ldots A_l}$ is a traceless symmetric tensor of $SO(3)$ with rank l [23]. The orthogonality condition for the spherical harmonics was given in equation (3.4.19). Similarly, $N \times N$ matrices of a matrix theory on a fuzzy sphere can be expanded as follows [23].

$$\begin{aligned} \hat{a} &= \sum_{l=0}^{N-1}\sum_{m=-l}^{l} a_{lm}\hat{Y}_{lm} && (5.1.13)\\ \hat{Y}_{lm} &= R^{-l}\sum_{\vec{A}} f^{(lm)}_{A_1\ldots A_l}\hat{x}^{A_1}\ldots\hat{x}^{A_l} && (5.1.14)\end{aligned}$$

by following the definition of the fuzzy sphere.[4] The matrices $\hat{Y}_{lm}$ are called fuzzy spherical harmonics. The spherical harmonics and hence the fuzzy spherical harmonics are tensor operators. The Wigner-Eckart theorem can be used to calculate combinations of fuzzy spherical harmonics, such as the orthogonality condition. The Wigner-Eckart theorem is [35, 37],

$$< j_1, m_1|\hat{Y}_{lm}|j_2, m_2 >= R_N(l)(-1)^{j_1-m_1}\begin{pmatrix} j_1 & l & j_2 \\ -m_1 & m & m_2\end{pmatrix} \tag{5.1.15}$$

where $(\ldots)$ is a Wigner 3j-symbol and $R_N(l)$ is a reduced matrix element. The orthogonality condition for the spherical harmonics involves an integration over the volume of the sphere. The analogous condition for matrices is the Trace [50].

$$\mathrm{Tr}_N\left(\hat{Y}^{\dagger}_{lm}\hat{Y}_{l'm'}\right) = \sum_{m_1,m_2} < j, m_1|Y^{\dagger}_{lm}|j, m_2 >< j, m_2|Y_{l'm'}|j, m_1 > \tag{5.1.16}$$

[4] $f^{(lm)}_{A_1\ldots A_l}$ is the same tensor as in (5.1.12).

Applying the Wigner-Eckart theorem and using the symmetries of the Wigner 3j-symbol [50],

$$\mathrm{Tr}_N\left(\hat{Y}^\dagger_{lm}\hat{Y}_{l'm'}\right) = R(l)R(l')\frac{1}{2l+1}\,\delta_{ll'}\delta_{mm'} \tag{5.1.17}$$

With the choice of normalisation $R(l) = \sqrt{2l+1}$ for the reduced matrix element [50], the orthogonality condition is,

$$\mathrm{Tr}_N\left(\hat{Y}^\dagger_{lm}\hat{Y}_{l'm'}\right) = \delta_{ll'}\,\delta_{mm'} \tag{5.1.18}$$

There is a clear relation between equations (3.4.21) and (5.1.13).

$$\hat{a} = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} a_{lm}\hat{Y}_{lm} \rightarrow\; a(\theta,\phi) = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi) \tag{5.1.19}$$

Notice that the expansion in spherical harmonics is truncated at $N-1$ reflecting the finite number of degrees of freedom in the matrix $\hat{a}$. This is a 1:1 mapping, formally given by [23],

$$a(\theta,\phi) = \sum_{lm}\mathrm{Tr}_N(\hat{Y}^\dagger_{lm}\hat{a})Y_{lm}(\theta,\phi) \tag{5.1.20}$$

Under this correspondence between matrices and functions, a matrix trace is equivalent to an integral over the 2-sphere [23, 24].

$$\frac{1}{N}\mathrm{Tr}_N \rightarrow \frac{1}{4\pi}\int d\Omega \tag{5.1.21}$$

The product of matrices maps to the star-product on the non-commutative sphere,[5]

$$a * b(\theta,\phi) = \sum_{lm}\mathrm{Tr}_N(\hat{Y}^\dagger_{lm}\hat{a}\hat{b})Y_{lm}(\theta,\phi) \tag{5.1.22}$$

This product is non-commutative due to the non-commutative nature of matrix multiplication [23]. This mapping produces a correspondence between matrix theories and non-commutative field theories.

[5]In order for the mapping to remain 1:1, it must be assumed that N is sufficiently large such that $l + l' \not> N - 1$.

In analogy to continuum field theory there are derivative operators for the matrix theory. They correspond to the adjoint action of $J_i^{(N)}$ [23].

$$Ad(J_3^{(N)}) = \sum_{lm} a_{lm}\left[J_3^{(N)}, \hat{Y}_{lm}\right] = \sum_{lm} a_{lm}\, m\, \hat{Y}_{lm} \tag{5.1.23a}$$

$$\begin{aligned} Ad(J_\pm^{(N)}) &= \sum_{lm} a_{lm}\left[J_\pm^{(N)}, \hat{Y}_{lm}\right] \\ &= \sum_{lm} a_{lm}\sqrt{(l \pm m + 1)(l \mp m)}\, \hat{Y}_{lm\pm 1} \end{aligned} \tag{5.1.23b}$$

The properties above, equations (5.1.23), show that by the correspondence between matrices and functions (5.1.19) the adjoint action of $J_i^{(N)}$ becomes [23],

$$Ad(J_i^{(N)}) \to L_i \tag{5.1.24}$$

The operator L_i is the derivative operator on the non-commutative sphere. Consequently, the fuzzy spherical harmonics do not commutate, satisfying the commutation relation [50],

$$[Y_{l_1 m_1}, Y_{l_2 m_2}] = F^{l_3 m_3}_{l_1 m_1 l_2 m_2} Y^\dagger_{l_3 m_3} \tag{5.1.25}$$

The $U(N)$ structure constants are,

$$\begin{aligned} F^{l_3 m_3}_{l_1 m_1 l_2 m_2} &= 2\sqrt{(2l_1+1)(2l_2+1)(2l_3+1)}(-1)^{N-1} \\ &\times \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \end{aligned} \tag{5.1.26}$$

where $\{\ldots\}$ is a Wigner 6j-symbol. For large-N the 6j-symbol behaves as $N^{-3/2}$ [50], so in the limit $N \to \infty$ the fuzzy spherical harmonics become commutative.

$$[Y_{l_1 m_1}, Y_{l_2 m_2}] = 0 \tag{5.1.27}$$

The usual commutative spherical harmonics are recovered in the limit $N \to \infty$, this is the commutative limit.

The M(atrix) theory takes a zero-dimensional matrix model and constructs a D-dimensional non-commutative field theory using the 1:1 correspondence between matrices and functions. These dimensions are physical, particles are able to propagate along them. The $\mathcal{N} = 1^*$ theory begins as a four-dimensional gauge theory whose Higgs vacuum describes a fuzzy sphere. The M(atrix) theory construction uses the correspondence between matrices and functions to construct a non-commutative, six-dimensional field theory on $\Re^{3,1} \times S^2_n$, where S^2_n denotes a non-commutative 2-sphere. The non-commutative sphere acts as a natural UV cut-off for the theory, by making it impossible to consider length scales smaller than the non-commutativity parameter, τ. In the limit $N \to \infty$ ($\tau \to 0$), the non-commutative sphere becomes commutative and the theory becomes a six-dimensional field theory on $\Re^{3,1} \times S^2$.

5.2 Higgsed $\mathcal{N} = 1^*$ SUSY Yang-Mills Theory

The six-dimensional theory originating from the $\mathcal{N} = 1^*$ theory can be identified through its classical spectrum and action. In this Section, the Lagrangian of the Higgsed $\mathcal{N} = 1^*$ theory will be constructed. Chapter 6 will then use this Lagrangian to calculate the classical spectrum of the Higgsed $\mathcal{N} = 1^*$ theory and construct its effective six-dimensional action. The construction proceeds by expanding the Lagrangian of the $U(N)$ $\mathcal{N} = 1^*$ theory of Section 2.6 (equation (2.6.5) shown below) about the Higgs vacuum.

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big\{ &-\frac{1}{4g_{ym}^2} F_{\mu\nu}F^{\mu\nu} - \frac{\Theta_{ym}}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{g_{ym}^2} i\lambda\sigma^\mu D_\mu\bar{\lambda} - i\psi_i\sigma^\mu D_\mu\bar{\psi}_i \\ &- D_\mu\Phi_i^\dagger D^\mu\Phi_i + \frac{1}{\sqrt{2}} i\psi_i[\Phi_i^\dagger, \lambda] - \frac{1}{\sqrt{2}} i\lambda[\Phi_i^\dagger, \psi_i] - \frac{1}{\sqrt{2}} i\bar{\lambda}[\Phi_i, \bar{\psi}_i] \\ &+ \frac{1}{\sqrt{2}} i\bar{\psi}_i[\Phi_i, \bar{\lambda}] - \frac{1}{2} g_{ym}^2[\Phi_i, \Phi_i^\dagger]^2 - g_{ym}^2[\Phi_j^\dagger, \Phi_i^\dagger][\Phi_i, \Phi_j] \\ &- \sqrt{2}\, ig_{ym}\eta\varepsilon_{ijk}[\Phi_i^\dagger, \Phi_j^\dagger]\Phi_k - \sqrt{2}\, ig_{ym}\eta\varepsilon_{ijk}\Phi_i^\dagger[\Phi_j, \Phi_k] - 4\eta^2\Phi_i^\dagger\Phi_i \\ &+ \frac{1}{\sqrt{2}} ig_{ym}\varepsilon_{ijk}\psi_i[\Phi_k, \psi_j] + \frac{1}{\sqrt{2}} ig_{ym}\varepsilon_{ijk}\bar{\psi}_i[\Phi_k^\dagger, \bar{\psi}_j] - \eta\psi_i\psi_i - \eta\bar{\psi}_i\bar{\psi}_i \Big\}\end{aligned} \tag{2.6.5}$$

The complex scalars are representations of the $SU(2)$ Lie algebra upon the reparameterisation,

$$\Phi_i \longrightarrow \frac{g_{ym}}{\sqrt{2}\eta}\,\Phi_i \tag{5.1.3}$$

The Higgs vacuum corresponds to the choice of vacuum $<\Phi_i>= J_i^{(N)}$. In addition to the reparameterisation (5.1.3), the Higgsed $\mathcal{N}=1^*$ Lagrangian will be simplified further by the reparameterisation of the remaining fields to make the Yang-Mills gauge coupling g_{ym} an overall coefficient and giving all fields a mass dimension of zero.

$$\psi_i \quad \longrightarrow \quad \frac{g_{ym}}{\sqrt{\eta^3}}\,\psi_i \tag{5.2.1a}$$

$$A_\mu \quad \longrightarrow \quad \frac{1}{\eta}\,A_\mu \tag{5.2.1b}$$

$$\lambda_i \quad \longrightarrow \quad \frac{1}{\sqrt{\eta^3}}\,\lambda_i \tag{5.2.1c}$$

After the reparameterisation of the fields (5.2.1) the Lagrangian of the $U(N)$ $\mathcal{N}=1^*$ theory is,

$$\begin{aligned}\mathcal{L} = \frac{1}{g_{ym}^2}\,\mathrm{Tr}\Big\{ & -\frac{1}{4}\eta^2 F_{\mu\nu}F^{\mu\nu} - i\eta^3\lambda\sigma^\mu D_\mu\bar{\lambda} - i\eta^3\psi_i\sigma^\mu D_\mu\bar{\psi}_i - 2\eta^2 D_\mu\Phi_i^\dagger D^\mu\Phi_i \\ & + \eta^4\Big(i\psi_i[\Phi_i^\dagger,\lambda] - i\lambda[\Phi_i^\dagger,\psi_i] - i\bar{\lambda}[\Phi_i,\bar{\psi}_i] + i\bar{\psi}_i[\Phi_i,\bar{\lambda}] - 2[\Phi_i,\Phi_i^\dagger]^2 \\ & - 4[\Phi_j^\dagger,\Phi_i^\dagger][\Phi_i,\Phi_j] - 4i\varepsilon_{ijk}[\Phi_i^\dagger,\Phi_j^\dagger]\Phi_k - 4i\varepsilon_{ijk}\Phi_i^\dagger[\Phi_j,\Phi_k] - 8\Phi_i^\dagger\Phi_i \\ & + i\varepsilon_{ijk}\psi_i[\Phi_k,\psi_j] + i\varepsilon_{ijk}\bar{\psi}_i[\Phi_k^\dagger,\bar{\psi}_j] - \psi_i\psi_i - \bar{\psi}_i\bar{\psi}_i\Big)\Big\}\end{aligned} \tag{5.2.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\eta[A_\mu, A_\nu]$, $D_\mu\phi = \partial_\mu\phi + i\eta[A_\mu,\phi]$ and the dual field strength term has been ignored. Note that in order for supersymmetry to be preserved during these reparameterisations, the supersymmetry transformations must be reparameterised appropriately.

The Lagrangian of the Higgsed $\mathcal{N}=1^*$ SUSY Yang-Mills theory is obtained by expanding the complex scalars about the Higgs vacuum,

$$\Phi_i \to \hat{\Phi}_i = J_i^{(N)} + \delta\hat{\Phi}_i$$

$\delta\hat{\Phi}_i$ are the matrix field fluctuations of the $\mathcal{N}=1^*$ theory. For the sake of clarity this process is displayed in four parts, beginning with the bosonic kinetic terms. The bosonic kinetic terms of the $\mathcal{N}=1^*$ theory are,

$$\mathcal{L}_{Bkin} = \eta^2\mathrm{Tr}\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2D_\mu\Phi_i^\dagger D^\mu\Phi_i\right\} \tag{5.2.3}$$

Under the expansion of the complex scalars $\Phi_i \to \hat{\Phi}_i = J_i^{(N)} + \delta\hat{\Phi}_i$.

$$\begin{aligned}\mathcal{L}_{Bkin} = \eta^2 \mathrm{Tr}_N \Big\{ & -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - 2\Big(\partial_\mu \delta\hat{\Phi}_i^\dagger \partial^\mu \delta\hat{\Phi}_i - i\eta[J_i, \hat{A}_\mu]\partial^\mu \delta\hat{\Phi}_i \\ & + i\eta[\hat{A}_\mu, \delta\hat{\Phi}_i^\dagger]\partial^\mu\delta\hat{\Phi}_i - i\eta\partial_\mu\delta\hat{\Phi}_i^\dagger[J_i, \hat{A}^\mu] + i\eta\partial_\mu\delta\hat{\Phi}_i^\dagger[\hat{A}^\mu, \delta\hat{\Phi}_i] \\ & - \eta^2[J_i, \hat{A}_\mu][J_i, \hat{A}^\mu] + \eta^2[J_i, \hat{A}_\mu][\hat{A}^\mu, \delta\hat{\Phi}_i] + \eta^2[\hat{A}_\mu, \delta\hat{\Phi}_i^\dagger][J_i, \hat{A}^\mu] \\ & - \eta^2[\hat{A}_\mu, \delta\hat{\Phi}_i^\dagger][\hat{A}^\mu, \delta\hat{\Phi}_i]\Big)\Big\}\end{aligned} \tag{5.2.4}$$

where $\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu + i\eta[\hat{A}_\mu, \hat{A}_\nu]$. Note that the trace over the gauge group in equation (5.2.3) becomes a trace over $N \times N$ matrices in (5.2.4), when the Higgs vacuum has been selected. Interactions between the bosons in the $\mathcal{N} = 1^*$ theory are described by the scalar potential.

$$\mathcal{V} = 4\eta^4 \mathrm{Tr}\, H_{ij}^\dagger H_{ij} + 2\eta^4 \mathrm{Tr}\, D^2 \tag{5.2.5}$$

where,

$$\begin{aligned} H_{ij} &= [\Phi_i, \Phi_j] - i\varepsilon_{ijk}\Phi_k \\ D &= [\Phi_i, \Phi_i^\dagger] \end{aligned}$$

Under the expansion of the complex scalars the objects H_{ij} and D become,

$$\begin{aligned} \hat{H}_{ij} &= [J_i, \delta\hat{\Phi}_j] - [J_j, \delta\hat{\Phi}_i] + [\delta\hat{\Phi}_i, \delta\hat{\Phi}_j] - i\varepsilon_{ijk}\delta\hat{\Phi}_k \\ \hat{D} &= [J_i, \delta\hat{\Phi}_i^\dagger] - [J_i, \delta\hat{\Phi}_i] + [\delta\hat{\Phi}_i^\dagger, \delta\hat{\Phi}_i] \end{aligned}$$

Consequently, under the expansion of the complex scalars, the scalar potential is,

$$\begin{aligned}\mathcal{V} = 4\eta^4 \mathrm{Tr}_N \Big\{ & -2[J_i, \delta\hat{\Phi}_j^\dagger][J_i, \delta\hat{\Phi}_j] + 2[J_i, \delta\hat{\Phi}_j^\dagger][J_j, \delta\hat{\Phi}_i] - 2[J_i, \delta\hat{\Phi}_j^\dagger][\delta\hat{\Phi}_i, \delta\hat{\Phi}_j] \\ & + 2i\varepsilon_{ijk}[J_i, \delta\hat{\Phi}_j^\dagger]\delta\hat{\Phi}_k - 2[\delta\hat{\Phi}_i^\dagger, \delta\hat{\Phi}_j^\dagger][J_i, \delta\hat{\Phi}_j] - [\delta\hat{\Phi}_i^\dagger, \delta\hat{\Phi}_j^\dagger][\delta\hat{\Phi}_i, \delta\hat{\Phi}_j] \\ & + i\varepsilon_{ijk}[\delta\hat{\Phi}_i^\dagger, \delta\hat{\Phi}_j^\dagger]\delta\hat{\Phi}_k + 2i\varepsilon_{ijk}\delta\hat{\Phi}_i^\dagger[J_j, \delta\hat{\Phi}_k] + i\varepsilon_{ijk}\delta\hat{\Phi}_i^\dagger[\delta\hat{\Phi}_j, \delta\hat{\Phi}_k] \\ & + 2\delta\hat{\Phi}_i^\dagger\delta\hat{\Phi}_i + \frac{1}{2}[J_i, \delta\hat{\Phi}_i^\dagger]^2 - [J_i, \delta\hat{\Phi}_i^\dagger][J_j, \delta\hat{\Phi}_j] + \frac{1}{2}[J_i, \delta\hat{\Phi}_i]^2 \\ & + [J_i, \delta\hat{\Phi}_i^\dagger][\delta\hat{\Phi}_j, \delta\hat{\Phi}_j^\dagger] - [J_i, \delta\hat{\Phi}_i][\delta\hat{\Phi}_j, \delta\hat{\Phi}_j^\dagger] + \frac{1}{2}[\delta\hat{\Phi}_i, \delta\hat{\Phi}_i^\dagger]^2 \Big\}\end{aligned} \tag{5.2.6}$$

The fermionic kinetic terms of the $\mathcal{N} = 1^*$ theory are,

$$\mathcal{L}_{Fkin} = -i\eta^3 \mathrm{Tr}\left\{ \lambda\sigma^\mu D_\mu \bar{\lambda} + \psi_i \sigma^\mu D_\mu \bar{\psi}_i \right\} \tag{5.2.7}$$

These terms do not contain the complex scalars Φ_i and are uneffected by the expansion of the complex scalars. The covariant derivative of the fermion kinetic terms can be expanded out,

$$\mathcal{L}_{Fkin} = -i\eta^3 \mathrm{Tr}_N\left\{ \hat{\lambda}\sigma^\mu \partial_\mu \hat{\bar{\lambda}} + i\eta\hat{\lambda}\sigma^\mu [\hat{A}_\mu, \hat{\bar{\lambda}}] + \hat{\psi}_i \sigma^\mu \partial_\mu \hat{\bar{\psi}}_i + i\eta\hat{\psi}_i \sigma^\mu [\hat{A}_\mu, \hat{\bar{\psi}}_i] \right\} \tag{5.2.8}$$

Finally interactions between the fermions and bosons of the chiral multiplets are described by the Yukawa potential.

$$\begin{aligned} \mathcal{L}_y = \eta^4 \mathrm{Tr}\Big\{ & i\psi_i \epsilon_{ijk}[\Phi_k, \psi_j] + i\bar{\psi}_i \epsilon_{ijk}[\Phi_k^\dagger, \bar{\psi}_j] - i\lambda[\Phi_i^\dagger, \psi_i] \\ & + i\psi_i[\Phi_i^\dagger, \lambda] + i\bar{\psi}_i[\Phi_i, \bar{\lambda}] - i\bar{\lambda}[\Phi_i, \bar{\psi}_i] - \psi_i\psi_i - \bar{\psi}_i\bar{\psi}_i \Big\} \end{aligned} \tag{5.2.9}$$

Under the expansion of the complex scalars,

$$\begin{aligned} \mathcal{L}_y = \eta^4 \mathrm{Tr}_N\Big\{ & i\hat{\psi}_i \epsilon_{ijk}[J_k, \hat{\psi}_j] + i\hat{\psi}_i \epsilon_{ijk}[\delta\hat{\Phi}_k, \hat{\psi}_j] + i\hat{\bar{\psi}}_i \epsilon_{ijk}[J_k, \hat{\bar{\psi}}_j] + i\hat{\bar{\psi}}_i \epsilon_{ijk}[\delta\hat{\Phi}_k^\dagger, \hat{\bar{\psi}}_j] \\ & - i\hat{\lambda}[J_i, \hat{\psi}_i] - i\hat{\lambda}[\delta\hat{\Phi}_i^\dagger, \hat{\psi}_i] + i\hat{\psi}_i[J_i, \hat{\lambda}] + i\hat{\psi}_i[\delta\hat{\Phi}_i^\dagger, \hat{\lambda}] + i\hat{\bar{\psi}}_i[J_i, \hat{\bar{\lambda}}] \\ & + i\hat{\bar{\psi}}_i[\delta\hat{\Phi}_i, \hat{\bar{\lambda}}] - i\hat{\bar{\lambda}}[J_i, \hat{\bar{\psi}}_i] - i\hat{\bar{\lambda}}[\delta\hat{\Phi}_i, \hat{\bar{\psi}}_i] - \hat{\psi}_i\hat{\psi}_i - \hat{\bar{\psi}}_i\hat{\bar{\psi}}_i \Big\} \end{aligned} \tag{5.2.10}$$

Chapter 6

Deconstruction of the Maldacena-Núñez Compactification

This Chapter will apply the deconstruction technique outlined in Section 5.1 to the Higgsed $\mathcal{N} = 1^*$ theory constructed in Section 5.2. It will begin with the calculation of the classical spectrum of the Higgsed $\mathcal{N} = 1^*$ theory in Section 6.1 and a direct comparison with the classical Kaluza-Klein spectrum of the Maldacena-Núñez compactified gauge theory. In Section 6.2 the effective six-dimensional action of the Higgsed $\mathcal{N} = 1^*$ theory will be derived and compared to the action constructed in Section 4.2.

6.1 Classical $\mathcal{N} = 1^*$ SUSY Yang-Mills Spectrum

In this Section the full classical spectrum of the Higgsed $\mathcal{N} = 1^*$ theory will be calculated. The spectrum is a list of all possibile particle states in a field theory and is split into a bosonic spectrum and a fermionic spectrum. In a supersymmetric theory the bosonic and fermionic spectrums should be identical with the particles forming supersymmetric multiplets. Only terms that are quadratic in the field fluctuations contribute to the mass spectrum, therefore the higher orders will be ignored during this calculation. The Lagrangian for the Higgsed $\mathcal{N} = 1^*$ theory was presented in Section 5.2.

The contribution to the mass spectrum is normalised by the kinetic terms of the fields. For example, if the Lagrangian of a free complex scalar field is,

$$\mathcal{L} = -a\partial_\mu \phi^\dagger \partial^\mu \phi - m^2|\phi|^2$$

then the square of the bosonic mass is $M^2 = \frac{m^2}{a}$. If the Lagrangian of a free spinor field is,

$$\mathcal{L} = -a\, i\psi\sigma^\mu D_\mu \bar{\psi} - \frac{1}{2}\, m\psi\psi - \frac{1}{2}\, m^* \bar{\psi}\bar{\psi}$$

then the square of fermionic mass is $M^2 = \frac{m^* m}{a^2}$.

In the Higgsed $\mathcal{N} = 1^*$ theory, the contribution to the fermionic mass spectrum comes from the Yukawa potential (5.2.10).

$$\begin{aligned}\mathcal{L}_{FM} = -2\eta\, \mathrm{Tr}_N \Big\{ & i\hat{\psi}_i\, \varepsilon_{ijk}[J_k, \hat{\psi}_j] + i\hat{\bar{\psi}}_i \varepsilon_{ijk}\, [J_k, \hat{\bar{\psi}}_j] - i\hat{\lambda}[J_i, \hat{\psi}_i] \\ & + i\hat{\psi}_i[J_i, \hat{\lambda}] + i\hat{\bar{\psi}}_i[J_i, \hat{\bar{\lambda}}] - i\hat{\bar{\lambda}}[J_i, \hat{\bar{\psi}}_i] - \hat{\psi}_i\hat{\psi}_i - \hat{\bar{\psi}}_i\hat{\bar{\psi}}_i \Big\}\end{aligned} \tag{6.1.1}$$

The label (N) that denotes the dimension of the matrix fields has been suppressed. This contribution is re-written to form the fermionic mass matrices Δ^{RS} and $\bar{\Delta}^{RS}$.

$$\mathcal{L}_{FM} = -2\eta \Big\{ \big(\hat{\Psi}_R\big)_{ab} \Delta^{(RS)}_{ab,cd} \big(\hat{\Psi}^T_S\big)_{cd} + \big(\hat{\bar{\Psi}}_R\big)_{ab} \bar{\Delta}^{(RS)}_{ab,cd} \big(\hat{\bar{\Psi}}^T_S\big)_{cd} \Big\} \tag{6.1.2}$$

The four species of Weyl fermions have been combined into a column vector $\hat{\Psi}_R$, with $\hat{\Psi}_R = \hat{\psi}_i$ for $R = i = 1, 2, 3$ and $\hat{\Psi}_4 = \hat{\lambda}$. The corresponding mass matrices Δ and $\bar{\Delta}$ are,

$$\Delta^{(ij)}_{ab,cd} = i\varepsilon_{ijk}\Big(\delta_{ac}(J_k)_{bd} - (J^*_k)_{ac}\,\delta_{bd}\Big) - \delta_{ij}\,\delta_{ac}\,\delta_{bd} = \bar{\Delta}^{(ij)}_{ab,cd} \tag{6.1.3a}$$

$$\Delta^{(i4)}_{ab,cd} = -i\Big((J^*_i)_{ac}\,\delta_{bd} - \delta_{ac}(J_i)_{bd}\Big) = \bar{\Delta}^{(i4)}_{ab,cd} \tag{6.1.3b}$$

$$\Delta^{(4i)}_{ab,cd} = -i\Big(\delta_{ac}(J_i)_{bd} - (J^*_i)_{ac}\,\delta_{bd}\Big) = \bar{\Delta}^{(4i)}_{ab,cd} \tag{6.1.3c}$$

The fermionic spectrum is derived from the square of the mass matrix.

$$M^{(RS)}_{ab,\,ef} = 4\eta^2 \bar{\Delta}^{(RT)}_{ab,\,cd}\, \Delta^{(TS)}_{cd,\,ef} \tag{6.1.4}$$

Ignoring the overall coefficient, the components of this matrix are,

$$\begin{aligned} M^{(ij)}_{ab,ef} &= (J^*_i J^*_j - J^*_j J^*_i)_{ae}\delta_{bf} + \delta_{ae}(J_i J_j - J_j J_i)_{bf} \\ &\quad -2i\varepsilon_{ijk}\Big\{\delta_{ae}(J_k)_{bf} - (J^*_k)_{ae}\delta_{bf}\Big\} + \delta_{ij}\Big\{\delta_{ae}(J_k J_k)_{bf} \\ &\quad -2(J^*_k)_{ae}(J_k)_{bf} + (J^*_k J^*_k)_{ae}\delta_{bf} + \delta_{ae}\delta_{bf}\Big\} \end{aligned} \tag{6.1.5a}$$

$$\begin{aligned} M^{(i4)}_{ab,ef} &= \varepsilon_{ijk}\Big\{(J^*_j)_{ae}(J_k)_{bf} - (J^*_k J^*_j)_{ae}\delta_{bf} - \delta_{ae}(J_k J_j)_{bf} \\ &\quad +(J^*_k)_{ae}(J_j)_{bf}\Big\} + i\Big\{(J^*_i)_{ae}\delta_{bf} - \delta_{ae}(J_i)_{bf}\Big\} \end{aligned} \tag{6.1.5b}$$

$$\begin{aligned} M^{(4i)}_{ab,ef} &= -\varepsilon_{ijk}\Big\{\delta_{ae}(J_j J_k)_{bf} - (J^*_k)_{ae}(J_j)_{bf} - (J^*_j)_{ae}(J_k)_{bf} \\ &\quad +(J^*_j J^*_k)_{ae}\delta_{bf}\Big\} - i\Big\{(J^*_i)_{ae}\delta_{bf} - \delta_{ae}(J_i)_{bf}\Big\} \end{aligned} \tag{6.1.5c}$$

$$M^{(44)}_{ab,ef} = \delta_{ae}(J_i J_i)_{bf} - 2(J^*_i)_{ae}(J_i)_{bf} + (J^*_i J^*_i)_{ae}\delta_{bf} \tag{6.1.5d}$$

The fermionic mass spectrum is given by the eigenvalues of the matrix $M^{(RS)}_{ab,\,ef}$.

The analysis presented in Section 5.1 demonstrated that the Higgs vacuum of the $\mathcal{N} = 1^*$ theory describes a fuzzy sphere. The matrix fields of the Higgsed $\mathcal{N} = 1^*$ theory can be expanded in their eigenstates, the fuzzy spherical harmonics, whose modes are labelled by the quantum numbers l, m. The calculation of the eigenvalues is greatly simplified by expanding the fermionic matrix fields in fuzzy spherical harmonics. First re-introduce the fermionic matrix fields by considering the bilinear form,

$$\mathcal{M}_F = \big(\hat{\Psi}^\dagger_R\big)_{ab} M^{(RS)}_{ab,\,ef}\big(\hat{\Psi}^T_S\big)_{ef} \tag{6.1.6}$$

which is analogous to a Lagrangian. Then the fermionic matrix fields $\hat{\Psi}_R$ are expanded in fuzzy spherical harmonics,

$$\hat{\Psi}_R = \sum_{lm} \Psi^{(R)}_{lm}\hat{Y}_{lm} \tag{6.1.7}$$

where $\Psi^{(R)}$ is a Grassman coefficient. By expanding the matrix fields in fuzzy spherical harmonics the bilinear form $\mathcal{M}_F$ becomes,

$$\mathcal{M}_F = 4\eta^2 \sum_{l=0}^{N-1}\sum_{m=-l}^{l}\sum_{l'=0}^{N-1}\sum_{m'=-l'}^{l'} \big(\Psi^{(R)}_{lm}\big)^\dagger \Psi^{(S)}_{l'm'} N^{(RS)}_{lm,\,l'm'} \tag{6.1.8}$$

The eigenvalues of the matrix $N^{(RS)}_{lm,l'm'}$ determine the fermionic mass spectrum.

$$N^{(RS)}_{lm,l'm'} = \delta_{ll'} \begin{pmatrix} J^2_{(L)}+1 & -iJ^{(L)}_3 & iJ^{(L)}_2 & 0 \\ iJ^{(L)}_3 & J^2_{(L)}+1 & -iJ^{(L)}_1 & 0 \\ -iJ^{(L)}_2 & iJ^{(L)}_1 & J^2_{(L)}+1 & 0 \\ 0 & 0 & 0 & J^2_{(L)} \end{pmatrix}_{mm'} \tag{6.1.9}$$

$L = 2l + 1$,

The bosonic mass spectrum receives a contribution from the scalar potential, which is the contribution from the complex scalar fields. Another contribution is received from the covariant derivative of the complex scalars, which is the contribution from the gauge bosons. To quadratic order in the complex scalars, the scalar potential of the Higgsed $\mathcal{N} = 1^*$ theory is,

$$\begin{aligned} \mathcal{V} = 2\eta^4 \mathrm{Tr}_N \Big\{ & -4[J_i, \delta\hat{\Phi}^\dagger_j][J_i, \delta\hat{\Phi}_j] + 4[J_i, \delta\hat{\Phi}^\dagger_j][J_j, \delta\hat{\Phi}_i] + 8i\varepsilon_{ijk}\delta\hat{\Phi}^\dagger_k[J_i, \delta\hat{\Phi}_j] \\ & + 4\delta\hat{\Phi}^\dagger_i\delta\hat{\Phi}_i + [J_i, \delta\hat{\Phi}^\dagger_i]^2 + 2[J_i, \delta\hat{\Phi}^\dagger_i][J_j, \delta\hat{\Phi}_j] + [J_i, \delta\hat{\Phi}_i]^2 \Big\} \end{aligned} \tag{6.1.10}$$

Gauge theories have sets of physically equivalent field configurations, which correspond to gauge transformations. In order to correctly calculate the mass spectrum for the bosons, the equivalent configurations must be eliminated otherwise the number of physical states will be over-counted. This is done by fixing the gauge. Physically equivalent field configurations correspond to degenerate vacua, which in a moduli space are the flat directions describing gauge transformations. The physically inequivalent configurations are orthogonal to the gauge transformations.[1]

$$Tr(\delta\Phi_i\, \delta_{gt}\Phi_i) = 0$$

An infinitesimal non-abelian gauge transformation with hermitian parameter λ is,

$$\delta_{gt}\Phi_i = i[\lambda, \Phi_i]$$

With this gauge transformation the orthogonality condition is,

$$Tr(i\lambda[\Phi_i, \delta\Phi_i]) = 0$$

[1]Many thanks to Tim Hollowood for the calculation of the gauge-fixing condition.

In the Higgs vacuum,

$$[\Phi_i, \delta\Phi_i] = [J_i^{(N)}, \delta\hat{\Phi}_i] \tag{6.1.11}$$

An appropriate gauge-fixing condition is $[J_i^{(N)}, \delta\hat{\Phi}_i] = 0$. From the cyclic property of the trace and the gauge condition $[J_i^{(N)}, \delta\hat{\Phi}_i] = 0$ it is found that the second term in equation (6.1.10) becomes,

$$\begin{aligned} \mathrm{Tr}_N[J_i, \delta\hat{\Phi}_j^\dagger][J_j, \delta\hat{\Phi}_i] &= \mathrm{Tr}_N\left([\delta\hat{\Phi}_j^\dagger, J_j][\delta\hat{\Phi}_i, J_i] - [\delta\hat{\Phi}_j^\dagger, \delta\hat{\Phi}_i][J_j, J_i]\right) \\ &= -\mathrm{Tr}_N\, i\varepsilon_{ijk}\delta\hat{\Phi}_k^\dagger[J_i, \delta\hat{\Phi}_j] \end{aligned} \tag{6.1.12}$$

The scalar potential is simplified by the application of the gauge condition.

$$\mathcal{V} = 8\eta^4\mathrm{Tr}_N\left\{\delta\hat{\Phi}_j^\dagger[J_i, [J_i, \delta\hat{\Phi}_j]] + i\varepsilon_{ijk}\delta\hat{\Phi}_i^\dagger[J_j, \delta\hat{\Phi}_k] + \delta\hat{\Phi}_i^\dagger\delta\hat{\Phi}_i\right\} \tag{6.1.13}$$

The contribution to the bosonic mass from the scalar potential is,

$$\mathcal{M}_V = 4\eta^2\mathrm{Tr}_N\left\{\delta\hat{\Phi}_j^\dagger[J_i, [J_i, \delta\hat{\Phi}_j]] + i\varepsilon_{ijk}\delta\hat{\Phi}_i^\dagger[J_j, \delta\hat{\Phi}_k] + \delta\hat{\Phi}_i^\dagger\delta\hat{\Phi}_i\right\} \tag{6.1.14}$$

The fluctuations in the complex scalars $\delta\hat{\Phi}_i$ can be expanded in fuzzy spherical harmonics,

$$\delta\hat{\Phi}_i = \sum_{l,m} \phi_{lm}^{(i)} \hat{Y}_{lm} \tag{6.1.15a}$$

$$\delta\hat{\Phi}_i^\dagger = \sum_{l,m} \bar{\phi}_{lm}^{(i)} \hat{Y}_{lm}^\dagger \tag{6.1.15b}$$

where $\phi_{lm}^{(i)}$ and $\bar{\phi}_{lm}^{(i)}$ are complex coefficients. Under the expansion in fuzzy spherical harmonics the contribution to the bosonic mass spectrum becomes,

$$\mathcal{M}_V = 4\eta^2 \sum_{l,m,l',m'} \left(\phi_{lm}^i\right)^\dagger \phi_{l'm'}^k N_{lm,l'm'}^{(ik)} \tag{6.1.16}$$

with the matrix,

$$N_{lm,l'm'}^{(ik)} = \left(J_{(L)}^2 + 1\right)_{mm'}\delta_{ll'}\delta_{ik} + i\varepsilon_{ijk}\left(J_j^{(L)}\right)_{mm'}\delta_{ll'} \tag{6.1.17}$$

The contribution to the bosonic mass spectrum from the gauge bosons comes from the covariant derivative of the complex scalars at quadratic order in the gauge bosons.

$$\mathcal{L}_D = 2\eta^4\mathrm{Tr}_N[J_i, \hat{A}_\mu][J_i, \hat{A}^\mu] \tag{6.1.18}$$

The contribution to the bosonic mass spectrum is,

$$\begin{aligned}\mathcal{M}_D = -4\eta^2\,\mathrm{Tr}_N[J_i,\hat{A}_\mu][J_i,\hat{A}^\mu] &= 4\eta^2\,\mathrm{Tr}_N\hat{A}_\mu[J^2,\hat{A}^\mu] \\ &= 4\eta^2 \sum_{lm,l'm'} a^\dagger_{(\mu)\,lm}\, a^{(\mu)}_{l'm'} \left(J^2_{(L)}\right)_{mm'} \delta_{ll'}\end{aligned} \tag{6.1.19}$$

In summary, the bilinear of the bosonic mass matrix is,

$$\mathcal{M}_B = \left(\hat{\Phi}^\dagger_R\right)_{ab} M^{(RS)}_{ab,ef} \left(\hat{\Phi}^T_S\right)_{ef} = 4\eta^2 \sum_{l=0}^{N-1}\sum_{m=-l}^{l}\sum_{l'=0}^{N-1}\sum_{m'=-l'}^{l'} \left(\phi^R_{lm}\right)^\dagger \phi^S_{l'm'} N^{(RS)}_{lm,l'm'} \tag{6.1.20}$$

where the fluctuations of the three complex scalars and the gauge boson are combined in a column vector $\hat{\Phi}_R$ with $\hat{\Phi}_R = \hat{\Phi}_i$ for $i = 1, 2, 3$ and $\hat{\Phi}_4 = \hat{A}_\mu$. The matrix $N^{(RS)}_{lm,l'm'}$ is exactly the same matrix as (6.1.9).

The fermionic and bosonic calculations lead to the same matrix. The mass spectrum is given by the eigenvalues of the matrix $N^{(RS)}_{lm,l'm'}$ and are determined by solving the characteristic equation. Consider the $(p+q)\times(p+q)$ matrix,

$$\mathcal{X} = \begin{array}{c c} & \begin{array}{cc} (p) & (q) \end{array} \\ \begin{array}{c} (p) \\ (q) \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \tag{6.1.21}$$

The determinant of $\mathcal{X}$ can be evaluated using the identity [51],

$$\det(\mathcal{X}) = \det(A)\det(D - CA^{-1}B) \tag{6.1.22}$$

The characteristic equation for the matrix $N^{(RS)}_{lm,l'm'}$ is,

$$\det\left(N - \lambda\mathbb{1}\right) = 0 \tag{6.1.23}$$

Applying the identity (6.1.22) to this characteristic equation,

$$\det\left(N - \lambda\mathbb{1}\right) = \det\left(\delta_{ll'}\right) \otimes \det\left((\gamma^{(L)} - 1)\delta_{mm'}\right) \det \begin{pmatrix} \gamma^{(L)}.\mathbb{1} & -iJ^{(L)}_3 & iJ^{(L)}_2 \\ iJ^{(L)}_3 & \gamma^{(L)}.\mathbb{1} & -iJ^{(L)}_1 \\ -iJ^{(L)}_2 & iJ^{(L)}_1 & \gamma^{(L)}.\mathbb{1} \end{pmatrix}_{mm'} \tag{6.1.24}$$

where $\gamma^{(L)} = l(l+1)+1-\lambda$ and λ is the eigenvalue of the characteristic equation. The gauge boson/gaugino contribution in the characteristic equation has decoupled from the complex scalar/chiral fermion contribution. Consequently, the eigenvalues of the gauge boson/gaugino contribution are trivial. The remaining non-trivial eigenvalues are calculated by evaluating the determinant of,

$$\tilde{N}_{mm'} = \begin{pmatrix} \gamma^{(L)}.\mathbb{1} & -iJ_3^{(L)} & iJ_2^{(L)} \\ iJ_3^{(L)} & \gamma^{(L)}.\mathbb{1} & -iJ_1^{(L)} \\ -iJ_2^{(L)} & iJ_1^{(L)} & \gamma^{(L)}.\mathbb{1} \end{pmatrix}_{mm'} \tag{6.1.25}$$

The matrix partition A is identified as,

$$A = \begin{pmatrix} \gamma^{(L)}.\mathbb{1} & -iJ_3^{(L)} \\ iJ_3^{(L)} & \gamma^{(L)}.\mathbb{1} \end{pmatrix} \tag{6.1.26}$$

The inverse of this matrix A^{-1} is,

$$A^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{6.1.27}$$

where,

$$a = \frac{\gamma^{(L)}}{(\gamma^{(l)})^2 - m^2}\,\delta_{mm'} \tag{6.1.28a}$$

$$b = \frac{im}{(\gamma^{(L)})^2 - m^2}\,\delta_{mm'} \tag{6.1.28b}$$

The determinant of $\tilde{N}$ is found to be,

$$\begin{aligned} \det \tilde{N}_{mm'} &= \prod_{m=-l}^{l} \frac{\big(\gamma^{(L)}-1\big)\big(\gamma^{(L)}-(l+1)\big)\big(\gamma^{(L)}+l\big)}{\big(\gamma^{(L)}-(m+1)\big)\big(\gamma^{(L)}+(m-1)\big)} \\ &= \big(\gamma^{(L)}-1\big)^{2l+1}\big(\gamma^{(L)}+l\big)^{2l+3}\big(\gamma^{(L)}-(l+1)\big)^{2l-1} \end{aligned} \tag{6.1.29}$$

With the determinant of $\tilde{N}$ calculated, the characteristic equation is evaluated to be,

$$\det\big(N-\lambda\mathbb{1}\big) = \prod_{l=0}^{N-1} (\gamma^{(L)}-1)^{2(2l+1)}(\gamma^{(L)}+l)^{2l+3}\big(\gamma^{(L)}-(l+1)\big)^{2l-1} \tag{6.1.30}$$

The roots of the characteristic equation yield the eigenvalues of the mass matrices.

Root	Eigenvalue
$\gamma^{(L)} - 1$	$l(l+1)$
$\gamma^{(L)} + l$	$(l+1)^2$
$\gamma^{(L)} - (l+1)$	l^2

The eigenvalues of the characteristic equation dictate the bosonic and fermionic mass spectrum.

The $l = 0$ states are,

Eigenvalue	Degeneracy
0	1
1	3

and the remaining $l = 1, 2, \dots, N-1$ states are,

Eigenvalue	Degeneracy
l^2	$2l - 1$
$l(l+1)$	$2(2l+1)$
$(l+1)^2$	$2l + 3$

These states form the complete spectrum, however it is more appropriate to sum over all values of an eigenvalue λ. The resulting spectrum contains a single zero eigenvalue and two series of eigenvalues labeled by a positive integer $k = 1, 2, \dots N-1$,

Eigenvalue	Degeneracy
k^2	$4k$
$k(k+1)$	$2(2k+1)$

There is an additional eigenvalue $\lambda = N^2$ with degeneracy $2N+1$. (Note: the mass of each state is given by the eigenvalues λ with the coefficient $4\eta^2$). As a final check the number of states can be counted and should be equal to the $4N^2$ states of the $U(N)$ $\mathcal{N} = 1^*$ theory.

$$\begin{aligned}\sum_{k=1}^{N-1} 4k + 2 &= 2N^2 - 2 \\ \sum_{k=1}^{N-1} 4k &= 2N^2 - 2N\end{aligned}$$

The total number of states is the sum of the number of states, for every eigenvalue.

$$\begin{aligned}\text{Total Number of States} &= \#(\lambda = 0) + \#(k(k+1)) + \#(k^2) + \#(N^2)\\ &= 1 + (2N^2 - 2) + (2N^2 - 2N) + 2N + 1\\ &= 4N^2\end{aligned}$$

Each eigenvalue of the bosonic matrix corresponds to a complex scalar or a gauge boson, whilst each eigenvalue of the fermionic matrix corresponds to a left-handed Weyl spinor and its right-handed charge conjugate. The fermions and bosons have an identical spectrum, as expected in a supersymmetric theory (the gauge bosons and gauginos having the eigenvalues $\sim l(l+1)$). The theory has $\mathcal{N} = 1$ supersymmetry, therefore the particle states must form $\mathcal{N} = 1$ supersymmetry multiplets. A $\mathcal{N} = 1$ supersymmetric theory containing particles of spin-1 or less can only form a chiral multiplet or a vector multiplet. A massless chiral multiplet has a spin-0 particle and a spin-$\frac{1}{2}$ particle; a massless vector multiplet has a spin-$\frac{1}{2}$ particle and a spin-1 particle (Table 2.2). A massive chiral multiplet has two spin-0 particles and a spin-$\frac{1}{2}$ particle; a massive vector multiplet has one spin-0 particle, two spin-$\frac{1}{2}$ particles and one spin-1 particle (Table 2.1). The $\mathcal{N} = 1^*$ theory has a $U(N)$ gauge symmetry which is broken to $U(1)$, therefore the spectrum must contain a single massless gauge boson (plus superpartner) and $N^2 - 1$ massive gauge bosons (plus superpartners) due to the Higgs mechanism.

The massless gauge boson and massless gaugino must form a massless vector multiplet and the massive gauge bosons and massive gauginos must form a massive vector multiplet. However, the massive gauge bosons and gauginos cannot form a massive vector multiplet alone, they must be accompanied by a massive spin-0 particle and another massive spin-$\frac{1}{2}$ particle with eigenvalue $l(l+1)$. The massive vector multiplet is constructed from a massless vector multiplet and a massless chiral multiplet (the supersymmetric analogue of a massless vector absorbing a massless scalar to form a massive vector in the Higgs mechanism). The remaining states are massive, consisting of spin-0 and spin-$\frac{1}{2}$ particles, therefore they must form massive chiral multiplets. In addition to a single massless vector multiplet, there is a 'Kaluza-Klein' tower of multiplets, labelled by $k = 1, 2, \ldots, N-1$,

M^2	Degeneracy	Multiplet
$4\eta^2 k(k+1)$	$2k+1$	Massive Vector
$4\eta^2 k^2$	$4k$	Massive Chiral

and $2N+1$ chiral multiplets with mass (squared) $M^2 = 4\eta^2 N^2$. In the contiuum limit $N \to \infty$, this spectrum is identical to the Kaulza-Klein spectrum of the Maldacena-Núñez compactified gauge theory with the identification $2\eta = \frac{1}{R}$. Furthermore, at finite N the spectrum of the Higgsed $\mathcal{N} = 1^*$ theory matches the Kaluza-Klein spectrum of the Maldacena-Núñez compactified gauge theory for states with mass less than $\sim N^2$ (with an additional $2N+1$ massive chiral multiplets of mass $\sim N^2$).

The classical spectrum can be generalised to the more general Higgs branches mentioned in Section 5.1. The vacua of these Higgs branches are,

$$< \hat{\Phi}_i >= \mathbb{1}_p \otimes J_i^{(q)} = \mathbb{1}_p \otimes \frac{1}{r}\,\hat{x}_i \tag{6.1.31}$$

The gauge group of the $\mathcal{N} = 1^*$ theory is broken,

$$U(N) = U(1) \times SU(p) \times SU(q) \to U(1) \times SU(p) = U(p) \tag{6.1.32}$$

A fuzzy sphere is formed by the $q \times q$ matrices. Matrices on the fuzzy sphere can be expanded in fuzzy spherical harmonics, so the fluctuations about the vacua can be expanded in fuzzy spherical harmonics,

$$\delta\hat{\Phi}_i = \sum_{lm} \phi^{(i)}_{lm(p)} \otimes \hat{Y}^{(q)}_{lm} \tag{6.1.33}$$

where the Fourier coefficients $\phi^{(i)}_{lm}$ are $p \times p$ matrices and the fuzzy spherical harmonics are $q \times q$ matrices. The calculation of the mass spectrum is identical to the $U(1)$ case except the that the Fourier coefficients are now $p \times p$ matrices, increasing the degeneracy.[2] The $k = 0$ mode describes a massless vector multiplet with degeneracy p^2. The remaining $q^2 - 1$ modes (at finite N) are,

[2]The classical spectrum originates from terms that are quadratic in the fields. These terms do not describe interactions between adjoint matter in a non-abelian gauge theory, therefore the spectrum will be identical to the free theory except for an increased degeneracy.

M^2	Degeneracy	Multiplet
$4\eta^2 k(k+1)$	$(2k+1)p^2$	Massive Vector
$4\eta^2 k^2$	$4k\,p^2$	Massive Chiral

for $k = 1, 2, \ldots, q-1$. The spectrum is completed by $(2q+1)p^2$ extra massive chiral multiplets with mass $M^2 = 4\eta^2 q^2$. The degeneracies of all states are integer multiples of p^2 which is consistent with each state transforming in the adjoint representation of the unbroken $U(p)$ gauge symmetry. The more general Higgs vacuum also allows the case of a $SU(N)$ gauge group to be considered. The general Higgs vacuum breaks the gauge group $SU(N) \to SU(p)$, in which case the degeneracy of each state is reduced from p^2 to $p^2 - 1$, which is appropriate for adjoint multiplets of $SU(p)$.

6.2 Effective Six-Dimensional Theory

In the previous Chapter it was shown that an effective six-dimensional field theory emerges in the Higgs vacuum of the $\mathcal{N} = 1^*$ theory. In the limit $N \to \infty$, the classical spectrum of the Higgsed $\mathcal{N} = 1^*$ theory is identical to the Kaluza-Klein spectrum of the Maldacena-Núñez compactified gauge theory. This is a clear indication that the effective six-dimensional theory is in fact the Maldacena-Núñez compactified gauge theory. In this Section the effective six-dimensional action of the Higgsed $\mathcal{N} = 1^*$ theory will be calculated, using the matrix-function correspondence of M(atrix) theory. This action will then be compared to the classical action of the Maldacena-Núñez compactified gauge theory. The starting point is to apply the deconstruction procedure outlined in Chapter 5 to the full action of the Higgsed $\mathcal{N} = 1^*$ theory presented in Section 5.2.[3] The four-dimensional matrix model is mapped to a six-dimensional non-commutative field theory on $\Re^{3,1} \times S^2_n$. In the limit $N \to \infty$, the non-commutative field theory becomes a commutative six-dimensional field theory on $\Re^{3,1} \times S^2$. The calculation is split into four parts for clarity; the scalar potential, the bosonic kinetic terms, the Yukawa potential and the fermionic kinetic terms.

The scalar potential with the gauge condition $[J_i, \delta\hat{\Phi}_i] = 0$ imposed has the 'action',

$$\begin{aligned} \mathcal{S}_V = 8\eta^4 \frac{1}{g^2_{ym}} \int d^4x \, \mathrm{Tr}_N \Big\{ & \delta\hat{\Phi}^\dagger_j [J_i, [J_i, \delta\hat{\Phi}_j]] + i\varepsilon_{ijk} \delta\hat{\Phi}^\dagger_i [J_j, \delta\hat{\Phi}_k] + \delta\hat{\Phi}^\dagger_i \delta\hat{\Phi}_i \\ & - [J_i, \delta\hat{\Phi}_j][\delta\hat{\Phi}^\dagger_j, \delta\hat{\Phi}_i] + [J_i, \delta\hat{\Phi}^\dagger_j][\delta\hat{\Phi}^\dagger_i, \delta\hat{\Phi}_j] + i\varepsilon_{ijk}[\delta\hat{\Phi}^\dagger_i, \delta\hat{\Phi}^\dagger_j]\delta\hat{\Phi}_k \\ & + i\varepsilon_{ijk}\delta\hat{\Phi}^\dagger_i [\delta\hat{\Phi}_j, \delta\hat{\Phi}_k] - [\delta\hat{\Phi}^\dagger_i, \delta\hat{\Phi}^\dagger_j][\delta\hat{\Phi}_i, \delta\hat{\Phi}_j] + \frac{1}{4}[\delta\hat{\Phi}^\dagger_i, \delta\hat{\Phi}_i]^2 \Big\} \end{aligned} \tag{6.2.1}$$

The correspondence between matrices and functions (equations (5.1.19) and (5.1.21)) states that in the non-commutative six-dimensional theory, the scalar potential has

[3] From now on it is more appropriate to discuss the action rather than the Lagrangian. In the deconstruction procedure the trace over the $N \times N$ matrices becomes an integral over the 2-sphere, consistent with an action rather than a Lagrangian.

become,

$$\begin{aligned}\mathcal{S}_V = 8\eta^4\frac{N}{4\pi g_{ym}^2}\int d^4x\int d\Omega\Big\{&\delta\Phi_i^\dagger\left(L^2\,\delta\Phi_i\right)+i\varepsilon_{ijk}\delta\Phi_i^\dagger\left(L_j\,\delta\Phi_k\right)+\delta\Phi_i^\dagger\delta\Phi_i\\&-(L_i\delta\Phi_j)[\delta\Phi_j^\dagger,\delta\Phi_i]+(L_i\delta\Phi_j^\dagger)[\delta\Phi_i^\dagger,\delta\Phi_j]+i\varepsilon_{ijk}[\delta\Phi_i^\dagger,\delta\Phi_j^\dagger]\delta\Phi_k\\&+i\varepsilon_{ijk}\delta\Phi_i^\dagger[\delta\Phi_j,\delta\Phi_k]-[\delta\Phi_i^\dagger,\delta\Phi_j^\dagger][\delta\Phi_i,\delta\Phi_j]+\frac{1}{4}[\delta\Phi_i^\dagger,\delta\Phi_i]^2\Big\}_\star\end{aligned}\tag{6.2.2}$$

where $\{\}_\star$ means all products are non-commutative star-products. In the commutative limit[4], the scalar potential is reduced to,

$$\mathcal{S}_V = 8\eta^4\frac{N}{4\pi g_{ym}^2}\int d^4x\int d\Omega\Big\{\delta\Phi_i^\dagger\left(L^2\,\delta\Phi_i\right)+i\varepsilon_{ijk}\delta\Phi_i^\dagger\left(L_j\,\delta\Phi_k\right)+\delta\Phi_i^\dagger\delta\Phi_i\Big\}\tag{6.2.3}$$

The bosonic action has the following kinetic terms,

$$\begin{aligned}\mathcal{S}_{Bkin} = \eta^2\frac{1}{g_{ym}^2}\int d^4x\,\mathrm{Tr}_N\Big\{&-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}-2\Big(\partial_\mu\delta\hat{\Phi}_i^\dagger\partial^\mu\delta\hat{\Phi}_i-i\eta[J_i,\hat{A}_\mu]\partial^\mu\delta\hat{\Phi}_i\\&+i\eta[\hat{A}_\mu,\delta\hat{\Phi}_i^\dagger]\partial^\mu\delta\hat{\Phi}_i-i\eta\partial_\mu\delta\hat{\Phi}_i^\dagger[J_i,\hat{A}^\mu]+i\eta\partial_\mu\delta\hat{\Phi}_i^\dagger[\hat{A}^\mu,\delta\hat{\Phi}_i]\\&-\eta^2[J_i,\hat{A}_\mu][J_i,\hat{A}^\mu]+\eta^2[J_i,\hat{A}_\mu][\hat{A}^\mu,\delta\hat{\Phi}_i]+\eta^2[\hat{A}_\mu,\delta\hat{\Phi}_i^\dagger][J_i,\hat{A}^\mu]\\&-\eta^2[\hat{A}_\mu,\delta\hat{\Phi}_i^\dagger][\hat{A}^\mu,\delta\hat{\Phi}_i]\Big)\Big\}\end{aligned}\tag{6.2.4}$$

where $\hat{F}_{\mu\nu}=\partial_\mu\hat{A}_\nu-\partial_\nu\hat{A}_\mu+i\eta[\hat{A}_\mu,\hat{A}_\nu]$ and $D_\mu\hat{\Phi}_i=\partial_\mu\hat{\Phi}_i+i\eta[\hat{A}_\mu,\hat{\Phi}_i]$. The correspondence between matrices and functions states that in the non-commutative field theory the bosonic kinetic terms are,

$$\begin{aligned}\mathcal{S}_{Bkin} = \frac{N}{4\pi g_{ym}^2}\eta^2\int d^4x\int d\Omega\Big\{&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-2\Big(\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i)\\&-i\eta\partial_\mu(\delta\Phi_i^\dagger)(L_iA^\mu)+i\eta\partial_\mu(\delta\Phi_i^\dagger)[A^\mu,\delta\Phi_i]-i\eta(L_iA_\mu)\partial^\mu(\delta\Phi_i)\\&+i\eta[A_\mu,\delta\Phi_i^\dagger]\partial^\mu(\delta\Phi_i)-\eta^2(L_iA_\mu)(L_iA^\mu)+\eta^2(L_iA_\mu)[A^\mu,\delta\Phi_i]\\&+\eta^2[A_\mu,\delta\Phi_i^\dagger](L_iA^\mu)+\eta^2[A_\mu,\delta\Phi_i^\dagger][A^\mu,\delta\Phi_i]\Big)\Big\}_\star\end{aligned}\tag{6.2.5}$$

[4]Note commutators of field fluctuations vanish in this limit, see equation (5.1.27), e.g. $[\delta\Phi_i,\delta\Phi_j]=0$

Taking the commutative limit and imposing the gauge condition $L_i\delta\Phi_i = 0$, the bosonic kinetic terms have been reduced to,

$$\begin{aligned}\mathcal{S}_{Bkin} = \frac{N}{4\pi g_{ym}^2}\eta^2 \int d^4x \int d\Omega\Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) \\ + 2\eta^2(L_iA_\mu)(L_iA^\mu)\Big\}\end{aligned} \tag{6.2.6}$$

where now $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The Yukawa potential of the Higgsed $\mathcal{N} = 1^*$ theory is,

$$\begin{aligned}\mathcal{S}_y = \eta^4\frac{1}{g_{ym}^2}\int d^4x\,\mathrm{Tr}_N\Big\{ i\hat{\psi}_i\varepsilon_{ijk}[J_k,\hat{\psi}_j] + i\hat{\psi}_i\varepsilon_{ijk}[\delta\hat{\Phi}_k,\hat{\psi}_j] + i\hat{\bar{\psi}}_i\varepsilon_{ijk}[J_k,\hat{\bar{\psi}}_j] \\ + i\hat{\bar{\psi}}_i\varepsilon_{ijk}[\delta\hat{\Phi}_k^\dagger,\hat{\bar{\psi}}_j] - i\hat{\lambda}[J_i,\hat{\psi}_i] - i\hat{\lambda}[\delta\hat{\Phi}_i^\dagger,\hat{\psi}_i] + i\hat{\psi}_i[J_i,\hat{\lambda}] + i\hat{\psi}_i[\delta\hat{\Phi}_i^\dagger,\hat{\lambda}] \\ + i\hat{\bar{\psi}}_i[J_i,\hat{\bar{\lambda}}] + i\hat{\bar{\psi}}_i[\delta\hat{\Phi}_i,\hat{\bar{\lambda}}] - i\hat{\bar{\lambda}}[J_i,\hat{\bar{\psi}}_i] - i\hat{\bar{\lambda}}[\delta\hat{\Phi}_i,\hat{\bar{\psi}}_i] - \hat{\psi}_i\hat{\psi}_i - \hat{\bar{\psi}}_i\hat{\bar{\psi}}_i\Big\}\end{aligned} \tag{6.2.7}$$

The correspondence between matrices and functions states that in the non-commutative field theory the Yukawa potential is,

$$\begin{aligned}\mathcal{S}_y = \frac{N}{4\pi g_{ym}^2}\eta^4\int d^4x\int d\Omega\Big\{ i\psi_i\varepsilon_{ijk}\left(L_k\psi_j\right) + i\psi_i\varepsilon_{ijk}[\delta\Phi_k,\psi_j] + i\bar{\psi}_i\varepsilon_{ijk}\left(L_k\bar{\psi}_j\right) \\ + i\bar{\psi}_i\varepsilon_{ijk}[\delta\Phi_k^\dagger,\bar{\psi}_j] - i\lambda\left(L_i\psi_i\right) - i\lambda[\delta\Phi_i^\dagger,\psi_i] + i\psi_i\left(L_i\lambda\right) + i\psi_i[\delta\Phi_i^\dagger,\lambda] \\ + i\bar{\psi}_i\left(L_i\bar{\lambda}\right) + i\bar{\psi}_i[\delta\Phi_i,\bar{\lambda}] - i\bar{\lambda}\left(L_i\bar{\psi}_i\right) - i\bar{\lambda}[\delta\Phi_i,\bar{\psi}_i] - \psi_i\psi_i - \bar{\psi}_i\bar{\psi}_i\Big\}_\star\end{aligned} \tag{6.2.8}$$

In the commutative limit the Yukawa potential has been reduced to,

$$\begin{aligned}\mathcal{S}_y = \frac{N}{4\pi g_{ym}^2}\eta^4\int d^4x\int d\Omega\Big\{ i\psi_i\,\varepsilon_{ijk}L_k\psi_j + i\bar{\psi}_i\,\varepsilon_{ijk}L_k\bar{\psi}_j - i\lambda L_i\psi_i \\ + i\psi_iL_i\lambda + i\bar{\psi}_iL_i\bar{\lambda} - i\bar{\lambda}L_i\bar{\psi}_i - \psi_i\psi_i - \bar{\psi}_i\bar{\psi}_i\Big\}\end{aligned} \tag{6.2.9}$$

The fermionic kinetic terms of the Higgsed $\mathcal{N} = 1^*$ theory are,

$$\begin{aligned}\mathcal{S}_{Fkin} = \eta^3\frac{1}{g_{ym}^2}\int d^4x\,\mathrm{Tr}_N\Big(-i\hat{\lambda}\sigma^\mu\partial_\mu\hat{\bar{\lambda}} + \eta\hat{\lambda}\sigma^\mu[\hat{A}_\mu,\hat{\bar{\lambda}}] \\ - i\hat{\psi}_i\sigma^\mu\partial_\mu\hat{\bar{\psi}}_i + \eta\hat{\psi}_i\sigma^\mu[\hat{A}_\mu,\hat{\bar{\psi}}_i]\Big)\end{aligned} \tag{6.2.10}$$

The correspondence between matrices and functions states that in the non-commutative field theory the fermionic kinetic terms are,

$$\mathcal{S}_{Fkin} = \frac{N}{4\pi g_{ym}^2}\eta^3 \int d^4x \int d\Omega\Big(- i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \eta\lambda\sigma^\mu[A_\mu, \bar{\lambda}] - i\psi_i\sigma^\mu\partial_\mu\bar{\psi}_i + \eta\psi_i\sigma^\mu[A_\mu, \bar{\psi}_i]\Big)_\star \tag{6.2.11}$$

In the commutative limit the fermionic kinetic terms have been reduced to,

$$\mathcal{S}_{Fkin} = \frac{N}{4\pi g_{ym}^2}\eta^3 \int d\Omega\Big(- i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - i\psi_i\sigma^\mu\partial_\mu\bar{\psi}_i\Big) \tag{6.2.12}$$

By the correspondence between matrices and functions, the effective six-dimensional action of the Higgsed $\mathcal{N} = 1^*$ theory has been found to be,

$$\begin{aligned}\mathcal{S} =& \frac{N}{4\pi g_{ym}^2}\eta^2 \int d^4x \int d\Omega\Big(- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) \\ &- i\eta\lambda\sigma^\mu\partial_\mu\bar{\lambda} - i\eta\psi_i\sigma^\mu\partial_\mu\bar{\psi}_i + \eta^2\Big\{ 2(L_iA_\mu)(L_iA^\mu) + i\psi_i\varepsilon_{ijk}L_k\psi_j \\ &+ i\bar{\psi}_i\varepsilon_{ijk}L_k\bar{\psi}_j - i\lambda L_i\psi_i + i\psi_iL_i\lambda + i\bar{\psi}_iL_i\bar{\lambda} - i\bar{\lambda}L_i\bar{\psi}_i - \psi_i\psi_i \\ &- \bar{\psi}_i\bar{\psi}_i - 8\delta\Phi_i^\dagger\left(L^2\,\delta\Phi_i\right) - 8i\varepsilon_{ijk}\delta\Phi_i^\dagger\left(L_j\,\delta\Phi_k\right) - 8\delta\Phi_i^\dagger\delta\Phi_i\Big\}\Big)\end{aligned} \tag{6.2.13}$$

The action can be rewritten in terms of Majorana spinors. The Majorana spinors of the chiral fermions and the gauginos are defined below.

$$\Psi_{iA} = \begin{pmatrix} \psi_{i\alpha} \\ \bar{\psi}_i^{\dot{\alpha}} \end{pmatrix} \qquad \Lambda_A = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}_i^{\dot{\alpha}} \end{pmatrix} \tag{6.2.14}$$

In terms of Majorana spinors the effective six-dimensional action is,

$$\begin{aligned}\mathcal{S} =& \frac{N}{4\pi g_{ym}^2}\eta^2 \int d^4x \int d\Omega\Big(- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) - \frac{i}{2}\eta\bar{\Lambda}\gamma^\mu\partial_\mu\Lambda \\ &- \frac{i}{2}\eta\bar{\Psi}_i\gamma^\mu\partial_\mu\Psi_i + \eta^2\Big\{ 2(L_iA_\mu)(L_iA^\mu) + i\bar{\Psi}_i\varepsilon_{ijk}L_k\Psi_j + 2i\bar{\Psi}_iL_i\Lambda \\ &- \bar{\Psi}_i\Psi_i - 8\delta\Phi_i^\dagger\left(L^2\,\delta\Phi_i\right) - 8i\varepsilon_{ijk}\delta\Phi_i^\dagger\left(L_j\,\delta\Phi_k\right) - 8\delta\Phi_i^\dagger\delta\Phi_i\Big\}\Big)\end{aligned} \tag{6.2.15}$$

The action (6.2.15) is manifestly six-dimensional, but it is not a canonical action of a six-dimensional field theory. The action retains the form of the four-dimensional $\mathcal{N} = 1^*$ theory, it is not manifestly Lorentz invariant. The Lorentz group for a six-dimensional field theory on $\Re^{5,1}$ is $SO(5,1)$. With two dimensions compactified on a 2-sphere the actual Lorentz group for this theory is the subgroup of $SO(3,1) \times SO(2)$. In order for the action to have a canonical form, the action (6.2.15) must be manifestly Lorentz invariant and there must be explicit kinetic terms for all six spacetime dimensions. The $SO(2)$ Lorentz subgroup is hidden in the $SO(6)$ R-symmetry of the $\mathcal{N} = 1^*$ theory and must be revealed.

The fields are functions of all six spacetime dimensions. In order to reconstruct the effective action as a canonical six-dimensional action, the degrees of freedom on the 2-sphere must be re-expressed appropriately. The degrees of freedom on the 2-sphere can be separated from the degrees of freedom on the 4-plane by performing a Fourier expansion on each field in terms of the eigenstates on the 2-sphere. The complex scalars $\delta\Phi_i$ form a 3-vector on the 2-sphere in the cartesian basis x^i. The analysis of vectors on the 2-sphere in Section 3.4.4 naively suggests that the Fourier expansion for the complex scalars is,

$$\delta\Phi_i(x,\theta,\phi) = \sum_{jm} \left(t_{jm} T_{(i)jm} + s_{jm} S_{(i)jm} + r_{jm} R_{(i)jm} \right) \tag{6.2.16}$$

The vector harmonics $T_{(i)jm}$ and $S_{(i)jm}$ are tangential to the 2-sphere whilst $R_{(i)jm}$ is normal to the 2-sphere. In [23] the author considered an expansion of the type,

$$\delta\Phi_i = K_i^a d_a + x_i \varphi \tag{6.2.17}$$

for a vector field d_a tangential to the 2-sphere,

$$d_a = \sum_{jm} \left(t_{jm} T_{jm\,a} + s_{jm} S_{jm\,a} \right)$$

and a scalar field φ normal to the 2-sphere.

$$\varphi = \sum_{jm} \varphi_{jm} Y_{jm}$$

This expansion posesses a striking similarity to (6.2.16), however close inspection reveals the expansion to be inconsistent and leads to a field theory with an incorrect mass spectrum.

To obtain a correct canonical effective six-dimensional action the expansion must preserve the mass spectrum of the $\mathcal{N} = 1^*$ theory (otherwise the action will describe a different theory). Such a requirement is indicative of a similarity transformation of the fields, using the eigenstates of the mass matrix to produce the correct action. By using this similarity transformation to change the basis, the fields will correspond directly to the mass eigenstates. The mass squared matrix $N^{(RS)}_{lm,l'm'}$ has a block diagonal form which simplifies the problem of determining the eigenstates. The eigenstates of the 3×3 operator matrix,

$$N^{(ij)} = \begin{pmatrix} L^2+1 & -iL_3 & iL_2 \\ iL_3 & L^2+1 & -iL_1 \\ -iL_2 & iL_1 & L^2+1 \end{pmatrix} \tag{6.2.18}$$

are found to be eigenstates of the 2-sphere, the vector harmonics: $Y^{(i)}_{llm}$ the corresponding eigenstate of $\lambda = l(l+1)$, $Y^{(i)}_{l+1,lm}$ the corresponding eigenstate to $\lambda = (l+1)^2$ and $Y^{(i)}_{l-1,lm}$ the corresponding eigenstate to $\lambda = l^2$; for integer $l \geq 0$. These eigenstates confirm that the naive expansion of the complex scalars $\delta\Phi_i$ (6.2.16) is a consistent expansion. The mass eigenstate of the trivial operator matrix,

$$N^{(4,4)} = L^2 \tag{6.2.19}$$

is the spherical harmonic Y_{lm}. A consistent expansion of the gauge potential would be,

$$A_\mu = \sum_{lm} A_{(\mu)lm} Y_{lm} \tag{6.2.20}$$

The vector harmonics and the spherical harmonic provide a complete orthonormal set of eigenstates.

Whilst the eigenstates above provide a consistent expansion of the effective action (6.2.15) they are not the most useful. The $SO(2)$ Lorentz subgroup of the effective six-dimensional theory remains hidden under a Fourier expansion in the vector harmonics.

In order for the action to have a canonical form, Lorentz invariance must be manifest. Therefore, before proceeding with the Fourier expansion, the $SO(2)$ Lorentz subgroup must be revealed. The scalar potential describes interactions in the Higgsed $\mathcal{N} = 1^*$ theory between the complex scalars.

$$\mathcal{V} = 8\eta^4 \int d\Omega \, \delta\Phi_i^\dagger \Delta_{ij} \, \delta\Phi_j \tag{6.2.21a}$$

$$\Delta_{ij} = (L^2 + 1)\delta_{ij} - i\varepsilon_{ijk} L_k \tag{6.2.21b}$$

The operator matrix Δ_{ij} is the bosonic mass matrix for the complex scalars. The scalar potential possesses a global $U(1)$ symmetry,

$$\delta\Phi_i \rightarrow e^{i\alpha}\delta\Phi_i \qquad \delta\Phi_i^\dagger \rightarrow e^{-i\alpha}\delta\Phi_i^\dagger \tag{6.2.22}$$

The $U(1)$ symmetry is a subgroup of the $SO(6)$ R-symmetry. The complex scalar fields can be expressed in terms of two real scalar fields,

$$\delta\Phi_i = \frac{1}{\sqrt{2}}\left(a_i + ib_i\right) \tag{6.2.23}$$

where a_i and b_i are real scalars. Conversely the real scalar fields can be expressed in terms of the complex scalar field and its hermitian conjugate.

$$a_i = \frac{1}{\sqrt{2}}\left(\delta\Phi_i + \delta\Phi_i^\dagger\right) \qquad ib_i = \frac{1}{\sqrt{2}}\left(\delta\Phi_i - \delta\Phi_i^\dagger\right) \tag{6.2.24}$$

The global $U(1)$ symmetry is converted to a $SO(2)$ symmetry by defining the 2-component object,

$$\mathcal{Y}_i^{\hat{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix} a_i \\ ib_i \end{pmatrix} \tag{6.2.25}$$

with the index $\hat{\alpha}$ labelling the two components. The $SO(2)$ subgroup of the Lorentz group has been revealed. In terms of this 2-component object the scalar potential is,

$$\mathcal{V} = 8\eta^4 \int d\Omega \, \mathcal{Y}_{i\hat{\alpha}}^\dagger (\hat{O}_{ij})^{\hat{\alpha}}{}_{\hat{\beta}} \mathcal{Y}_j^{\hat{\beta}}$$

where the matrix $(\hat{O}_{ij})^{\hat{\alpha}}{}_{\hat{\beta}} = \delta^{\hat{\alpha}}{}_{\hat{\beta}}\Delta_{ij}$. The 2-component object $\mathcal{Y}_i$ must be expanded in the eigenstates of the operator $(\hat{O}_{ij})$, the mass matrix for $\mathcal{Y}_i$. One also wants the expansion to be a Fourier expansion in the eigenstates of the 2-sphere. From the treatment of the eigenstates of the 2-sphere presented in Section 3.4, it is found that the complete set of eigenstates of the operator $\hat{O}_{ij}$ are,

$$e_i^{\hat{\alpha}} = v^{\hat{\alpha}} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm}(\theta,\phi) \tag{6.2.26a}$$

$$\chi_{i\pm}^{\hat{\alpha}} = (\sigma_i)^{\hat{\alpha}}{}_{\hat{\beta}} \Omega^{\hat{\beta}}_{q_\pm lm}(\theta,\phi) + \frac{1}{\kappa_\pm} L_i \Omega^{\hat{\alpha}}_{q_\pm lm}(\theta,\phi) \tag{6.2.26b}$$

where $v^{\hat{\alpha}}$ is an arbitrary 2-component object. Under the action of the operator $\hat{O}_{ij}$,

$$\hat{O}_{ij}\, e_j^{\hat{\alpha}} = l(l+1) e_i^{\hat{\alpha}} \tag{6.2.27a}$$

$$\hat{O}_{ij}\, \chi_{j\pm}^{\hat{\alpha}} = \kappa_\pm^2 \chi_{i\pm}^{\hat{\alpha}} \tag{6.2.27b}$$

To construct a canonical action the 2-component object $\mathcal{Y}_i^{\hat{\alpha}}$ is expanded in the complete basis of eigenstates,

$$\mathcal{Y}_i^{\hat{\alpha}} = \mathcal{A}_i^{\hat{\alpha}} + \mathcal{P}_i^{\hat{\alpha}} \tag{6.2.28}$$

with,

$$\mathcal{A}_i^{\hat{\alpha}} = \sum_{lm} v_{lm}^{\hat{\alpha}} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm}(\theta,\phi) \tag{6.2.29a}$$

$$\begin{aligned} \mathcal{P}_i^{\hat{\alpha}} = \sum_{lm} \Bigg\{ & \xi^+_{lm} \left((\sigma_i)^{\hat{\alpha}}{}_{\hat{\beta}} \Omega^{\hat{\beta}}_{q_+ lm} + \frac{1}{\kappa_+} L_i \Omega^{\hat{\alpha}}_{q_+ lm} \right) \\ & + \xi^-_{lm} \left((\sigma_i)^{\hat{\alpha}}{}_{\hat{\beta}} \Omega^{\hat{\beta}}_{q_- lm} + \frac{1}{\kappa_-} L_i \Omega^{\hat{\alpha}}_{q_- lm} \right) \Bigg\} \end{aligned} \tag{6.2.29b}$$

$v_{lm}^{\hat{\alpha}}$ is an arbitrary 2-component object for each value $\{l,m\}$ and $\xi_{lm}^{(\pm)}$ is a complex coefficient. For a consistent expansion, the number of degrees of freedom must be conserved. The 2-component object $\mathcal{Y}_i^{\hat{\alpha}}$ has two real degrees of freedom. The object $\mathcal{A}_i^{\hat{\alpha}}$ has four real degrees of freedom: a spherical harmonic has two real degrees of freedom for each $\{l,m\}$ and the 2-component object $v_{lm}^{\hat{\alpha}}$ has two real degrees of freedom, one for each value of $\hat{\alpha}$ for a given $\{l,m\}$. The object $\mathcal{P}_i^{\hat{\alpha}}$ has two real degrees of freedom: the spherical spinor has two real degrees of freedom for each $\{l,m\}$. The expansion is subject to a constraint, the gauge-fixing condition.

$$L_i \delta\Phi_i = \frac{1}{\sqrt{2}} L_i (a_i + ib_i) = 0 \tag{6.2.30}$$

Consider imposing the gauge-fixing condition on the two-component object $\mathcal{Y}_i^{\hat{\alpha}}$. Denote the two components of $v_{lm}^{\hat{\alpha}}$ as,

$$v_{lm}^{\hat{\alpha}} = \begin{pmatrix} y_{lm} \\ z_{lm} \end{pmatrix}$$

The fields a_i and b_i can be identified as,

$$\begin{aligned} \frac{1}{\sqrt{2}} a_i &= \sum_{lm} \left\{ y_{lm} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm} + \sum_{+,-} \xi_{lm}^{(\pm)} \left(\sigma_i \Omega_{q_\pm lm} + \frac{1}{\kappa_\pm} L_i \Omega_{q_\pm lm} \right)^{\hat{1}} \right\} \\ \frac{1}{\sqrt{2}} ib_i &= \sum_{lm} \left\{ z_{lm} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm} + \sum_{+,-} \xi_{lm}^{(\pm)} \left(\sigma_i \Omega_{q_\pm lm} + \frac{1}{\kappa_\pm} L_i \Omega_{q_\pm lm} \right)^{\hat{2}} \right\} \end{aligned}$$

The T-spinor term is not constrained by the gauge-fixing condition,

$$L_i \left(\sigma_i \Omega_{q_\pm lm} + \frac{1}{\kappa_\pm} L_i \Omega_{q_\pm lm} \right)^{\hat{\alpha}} = 0 \tag{6.2.31}$$

so no constraint is imposed on $\xi_{lm}^{(\pm)}$. The remaining terms are,

$$\begin{aligned} L_i \delta \Phi_i &= L_i \left(\sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm} + \sum_{lm} z_{lm} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm} \right) \\ &= \sum_{lm} (y_{lm} + z_{lm}) \frac{1}{\sqrt{l(l+1)}} iL^2 Y_{lm} = 0 \end{aligned}$$

The gauge-fixing condition imposes the constraint $y_{lm} = -\, z_{lm}$ and allows a choice of gauge,

$$v_{lm}^{\hat{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} y_{lm} = v^{\hat{\alpha}}\, y_{lm} \tag{6.2.32}$$

so that $v_{\hat{\alpha}}^{\dagger} v^{\hat{\alpha}} = 1$. By fixing the gauge the object $\mathcal{A}_i^{\hat{\alpha}}$ has only two real degrees of freedom.

$$\mathcal{A}_i^{\hat{\alpha}} = v^{\hat{\alpha}} \sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm} \tag{6.2.33}$$

The scalar field $a_i = (a_i)^*$ is real. This implies a reality condition on the complex coefficient y_{lm}.

$$\begin{aligned}\sum_{lm} y_{lm}\, iL_i Y_{lm} &= \sum_{lm} y^*_{lm} iL_i Y^*_{lm} \\ &= \sum_{lm} y^*_{lm}\, iL_i(-1)^m Y_{l,-m} \\ &= \sum_{lm} y^*_{l,-m}\,(-1)^{-m} iL_i Y_{lm}\end{aligned}$$

Under complex conjugation the complex coefficient transforms as,

$$y^*_{lm} = (-1)^{-m} y_{l,-m} \tag{6.2.34}$$

The eigenstates of $\hat{O}_{ij}$ are all orthogonal, therefore the cross-terms of the eigenstates in an expansion are zero. Suppressing the indices $\hat{\alpha}$, the scalar potential is,

$$\mathcal{V} = 8\eta^4 \int d\Omega \left\{ \mathcal{A}_i^\dagger \hat{O}_{ij} \mathcal{A}_j + \mathcal{P}_i^\dagger \hat{O}_{ij} \mathcal{P}_j \right\} \tag{6.2.35}$$

The expansion of the first term in the scalar potential gives,

$$\int d\Omega\, \mathcal{A}_i^\dagger \hat{O}_{ij} \mathcal{A}_j = R^4 \int d\Omega \sum_{lm,l'm'} y^\dagger_{lm}\, y_{l'm'} \frac{1}{\sqrt{l(l+1)l'(l'+1)}} Y^\dagger_{lm} \Delta^2_{S^2} Y_{l'm'} \tag{6.2.36}$$

where the Laplacian on the 2-sphere is,

$$\Delta_{S^2} = \frac{1}{\sqrt{g}} \partial_a \left(g^{ab} \sqrt{g}\, \partial_b \right) = \frac{1}{R^2} \left(\cot\theta\, \partial_\theta + \partial_\theta \partial_\theta + \csc^2\theta\, \partial_\phi \partial_\phi \right) \tag{6.2.37}$$

In Section 3.4.4 it was shown that vectors on the 2-sphere can be expanded in terms of the covariant and contravariant vector harmonics, $T_{lm\,a}$ and $T_{lm}{}^a$. From the definition of these vector harmonics, vector fields on the 2-sphere can be defined from the spherical harmonics,

$$n_\theta(\theta,\phi) = R \sum_{lm} y_{lm} \frac{(-1)}{\sqrt{l(l+1)}} \csc\theta\, \partial_\phi Y_{lm}(\theta,\phi) \tag{6.2.38a}$$

$$n_\phi(\theta,\phi) = R \sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}} \sin\theta\, \partial_\theta Y_{lm}(\theta,\phi) \tag{6.2.38b}$$

The hermiticity condition of y_{lm} means the T-vector fields n_a are also real.

$$\begin{aligned} n_\theta^*(\theta,\phi) &= R\sum_{lm} y_{lm}^* \frac{(-1)}{\sqrt{l(l+1)}}\csc\theta\,\partial_\phi Y_{lm}^*(\theta,\phi) \\ &= R\sum_{lm}(-1)^{-m} y_{l,-m}\frac{(-1)}{\sqrt{l(l+1)}}\csc\theta\,(-1)^m\,\partial_\phi Y_{l,-m}(\theta,\phi) \\ &= n_\theta(\theta,\phi) \end{aligned} \tag{6.2.39}$$

Equation (6.2.36) can be expressed in terms of the vector field n_a.

$$\begin{aligned} R^2\sum_{lm} y_{lm}\frac{1}{\sqrt{l(l+1)}}\Delta_{S^2}Y_{lm} &= \sum_{lm} y_{lm}\frac{1}{\sqrt{l(l+1)}}\Big\{\csc\theta\,\partial_\theta(\sin\theta\,\partial_\theta Y_{lm}) \\ &\qquad + \csc\theta\,\partial_\phi(\csc\theta\,\partial_\phi Y_{lm})\Big\} \\ &= \frac{1}{R}\left(\csc\theta\,\partial_\theta n_\phi - \csc\theta\,\partial_\phi n_\theta\right) \\ &= \frac{1}{R}\csc\theta\,\mathcal{F}_{\theta\phi} \end{aligned}$$

where $\mathcal{F}_{\theta\phi} = \partial_\theta n_\phi - \partial_\phi n_\theta$. In terms of the T-vector fields n_a, the first term of the scalar potential (equation (6.2.36)) becomes,

$$\begin{aligned} 8\eta^4\int d\Omega\,\mathcal{A}_i^\dagger \hat{O}_{ij}\mathcal{A}_j &= 8\eta^4\frac{1}{R^2}\int d\Omega\,\csc^2\theta\mathcal{F}_{\theta\phi}\mathcal{F}_{\theta\phi} \\ &= \eta^2\int d\Omega\,\mathcal{F}_{ab}\mathcal{F}^{ab} \end{aligned} \tag{6.2.40}$$

using the identification $4\eta^2 = \frac{1}{R^2}$. This term describes the propagation of the bosonic T-vector n_a on the 2-sphere.

The expansion of the second term in the scalar potential (6.2.35) is,

$$\begin{aligned} \int d\Omega\,\mathcal{P}_i^\dagger\hat{O}_{ij}\mathcal{P}_j &= \int d\Omega\,\mathcal{P}_i^\dagger\sum_{l'm'}\sum_{+,-}\xi_{l'm'}^{(\pm)}\,\hat{O}_{ij}\left(\sigma_j\Omega_{q'_\pm l'm'} + \frac{1}{\kappa'_\pm}L_j\Omega_{q'_\pm l'm'}\right) \\ &= R^2\int d\Omega\,\mathcal{P}_i^\dagger\sum_{l'm'}\sum_{+,-}\xi_{l'm'}^{(\pm)}\left(\sigma_i\kappa^2\Omega_{q'_\pm l'm'} + \frac{1}{\kappa'_\pm}L_i\kappa^2\Omega_{q'_\pm l'm'}\right) \\ &= R^2\int d\Omega\sum_{lm,l'm'}\sum_{+,-}\left(\xi_{lm}^{(\pm)}\right)^\dagger\xi_{l'm'}^{(\pm)}\,\Omega_{q_\pm lm}^\dagger\left(3\kappa^2 + \frac{1}{\kappa'_\pm}\sigma_i L_i\,\kappa^2\right. \\ &\qquad \left. + \frac{1}{\kappa_\pm}\sigma_i L_i\,\kappa^2 + \frac{1}{\kappa_\pm\,\kappa'_\pm}L^2\kappa^2\right)\Omega_{q'_\pm l'm'} \end{aligned} \tag{6.2.41}$$

The cross-terms such as,

$$\int d\Omega\, \Omega^{\dagger}_{q_+ lm} 3\kappa^2 \Omega_{q'_- l'm'}$$

vanish due to orthogonality of the spherical spinors of different q, q'. The eigenstates $\mathcal{P}_i$ have not been normalised (unlike $\mathcal{A}_i$). The eigenstates can be normalised by a redefinition of the complex coefficient $\xi^{(\pm)}_{lm}$. Consider the expression,

$$\int d\Omega\, \Omega^{\dagger}_{q_\pm lm} \left(3 + \frac{1}{\kappa'_\pm}\sigma_i L_i + \frac{1}{\kappa_\pm}\sigma_i L_i + \frac{1}{\kappa_\pm \kappa'_\pm} L^2\right) \Omega_{q'_\pm l'm'}$$

Due to the orthogonality condition of the spherical spinors (3.4.40) this can be written as,

$$\int d\Omega\, \Omega^{\dagger}_{q_\pm lm}\left(3 - \frac{2}{\kappa'_\pm}(\kappa'_\pm + 1) + \frac{1}{\kappa'^2_\pm} L^2\right)\Omega_{q'_\pm l'm'} = \int d\Omega\, \Omega^{\dagger}_{q_\pm lm}\left(2 - \frac{1}{\kappa'_\pm}\right)\Omega_{q'_\pm l'm'}$$

The eigenstates are normalised by re-defining the complex coefficient,

$$\xi^{(\pm)}_{lm} \longrightarrow \sqrt{2 - \frac{1}{\kappa_\pm}}\, \xi^{(\pm)}_{lm} \tag{6.2.42}$$

Applying this re-definition to the second term in the scalar potential,

$$\begin{aligned} 8\eta^4 \int d\Omega\, \mathcal{P}^{\dagger}_i \hat{O}_{ij} \mathcal{P}_j &= 8\eta^4 R^2 \int d\Omega \sum_{lm,l'm'} \sum_{+,-} \left(\xi^{(\pm)}_{lm}\right)^{\dagger} \xi^{(\pm)}_{l'm'}\, \Omega^{\dagger}_{q_\pm lm}\, \kappa^2 \Omega_{q'_\pm l'm'} \\ &= 2\eta^2 \int d\Omega\, \xi^{\dagger}_{\hat\alpha}(\theta,\phi)\, \left(\kappa^2\right)^{\hat\alpha}{}_{\hat\beta}\, \xi^{\hat\beta}(\theta,\phi) \end{aligned} \tag{6.2.43}$$

where the 2-component object,

$$\xi^{\hat\alpha}(\theta,\phi) = \sum_{lm}\sum_{+,-} \xi^{(\pm)}_{lm} \Omega^{\hat\alpha}_{q_\pm lm}(\theta,\phi) \tag{6.2.44}$$

is a T-spinor field. The term describes the propagation of the T-spinor on the 2-sphere. In summary, after expanding the 2-component object $\mathcal{Y}^{\hat\alpha}_i$ in eigenstates on the 2-sphere, the scalar potential's contibution to the action is,

$$\mathcal{S}_V = \frac{N}{4\pi g^2_{ym}}\eta^2 \int d^4x \int d\Omega \left\{\mathcal{F}_{ab}\mathcal{F}^{ab} + 2\,\xi^{\dagger}_{\hat\alpha}(\kappa^2)^{\hat\alpha}{}_{\hat\beta}\xi^{\hat\beta}\right\} \tag{6.2.45}$$

The remaining bosonic terms of the effective action (6.2.15) originated from the kinetic terms of the $\mathcal{N} = 1^*$ theory. The kinetic terms describe the propagation of the complex scalars and gauge bosons in the Higgsed $\mathcal{N} = 1^*$ theory and they also describe the interactions between the complex scalars and gauge bosons.

$$\begin{aligned} \mathcal{S}_{Bkin} = \frac{N}{4\pi g_{ym}^2}\eta^2 \int d^4x \int d\Omega \Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) \\ + 2\eta^2(L_iA_\mu)(L_iA^\mu)\Big\} \end{aligned} \tag{6.2.46}$$

The gauge boson is expanded in spherical harmonics,

$$A_\mu(\theta,\phi) = \sum_{lm} A_{(\mu)lm}Y_{lm}(\theta,\phi) \tag{6.2.47}$$

where $A_{(\mu)lm}$ is a complex coefficient. The field tensor expands in a trivial manner,

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \sum_{lm}\left(\partial_\mu A_{(\nu)lm} - \partial_\nu A_{(\mu)lm}\right)Y_{lm} \end{aligned} \tag{6.2.48}$$

Therefore the gauge kinetic term, describing the propagation of the gauge bosons on the 4-plane is unchanged. The third term in (6.2.46) expands in a straight forward manner.

$$2\eta^2 \int d\Omega(L_iA_\mu)(L_iA^\mu) = 2\eta^2R^2 \int d\Omega\ A_\mu\Delta_{S^2}A^\mu = \frac{1}{2}\int d\Omega\ A_\mu\Delta_{S^2}A^\mu \tag{6.2.49}$$

This term describes the propagation of the gauge boson A_μ (a T-scalar) on the 2-sphere. The second term in (6.2.46) is the kinetic term for the complex scalars. The first step is to construct the 2-component object $\mathcal{Y}_i$.

$$2\eta^2 \int d\Omega\, \partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) = 2\eta^2 \int d\Omega\ \partial_\mu\mathcal{Y}_i^\dagger\partial^\mu\mathcal{Y}_i$$

The 2-component object is then expanded in the eigenstates of $\hat{O}_{ij}$.

$$2\eta^2 \int d\Omega\ \partial_\mu\mathcal{Y}_i^\dagger\partial^\mu\mathcal{Y}_i = 2\eta^2 \int d\Omega\ \left(\partial_\mu\mathcal{A}_i^\dagger\partial^\mu\mathcal{A}_i + \partial_\mu\mathcal{P}_i^\dagger\partial^\mu\mathcal{P}_i\right)$$

The following procedure is basically the same as was followed for the scalar potential.

$$2\eta^2 \int d\Omega\, \partial_\mu \mathcal{A}_i^\dagger \partial^\mu \mathcal{A}_i = 2\eta^2 \int d\Omega \sum_{lm} \sum_{l'm'} \partial_\mu y_{lm}^\dagger\, \partial^\mu y_{l'm'} \frac{1}{\sqrt{l(l+1)l'(l'+1)}}\, Y_{lm}^\dagger L^2 Y_{l'm'} \tag{6.2.50}$$

From equation (3.4.7) and the definition of the vector harmonic $T_{lm\,a}$ equation (3.4.69a), this term can be expressed in terms of the vector field n_a. First,

$$\sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}}\, iL_i\, Y_{lm} = \sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}}\, K_i^a \partial_a Y_{lm} = \frac{1}{R}\frac{1}{\sqrt{g}}\, K_i^a g_{ab}\, \varepsilon^{bc} n_c$$

Substituting this expression into the term,

$$\begin{aligned} 2\eta^2 \int d\Omega\, \partial_\mu \mathcal{A}_i^\dagger \partial^\mu \mathcal{A}_i &= 2\eta^2 \frac{1}{R^2} \int d\Omega\, \partial_\mu \left(\frac{1}{\sqrt{g}}\, K_i^a g_{ab}\, \varepsilon^{bc} n_c \right) \partial^\mu \left(\frac{1}{\sqrt{g}}\, K_i^d g_{de}\, \varepsilon^{ef} n_f \right) \\ &= 2\eta^2 \int d\Omega\, \frac{1}{g}\, \varepsilon^{bc} g_{be}\, \varepsilon^{ef} \partial_\mu n_c\, \partial^\mu n_f \\ &= 2\eta^2 \int d\Omega\, \partial_\mu n_a \partial^\mu n^a \end{aligned} \tag{6.2.51}$$

This term describes the propagation of the T-vector n_a on the 4-plane. The final term of the bosonic kinetic terms is found to be,

$$\begin{aligned} 2\eta^2 \int d\Omega\, \partial_\mu \mathcal{P}_i^\dagger \partial^\mu \mathcal{P}_i &= 2\eta^2 \int d\Omega \sum_{lm,l'm'} \sum_{+,-} \partial_\mu (\xi_{lm}^{(\pm)})^\dagger \partial^\mu \xi_{l'm'}^{(\pm)} \left(\Omega_{q_\pm lm}^\dagger \sigma_i + \frac{1}{\kappa_\pm} \Omega_{q_\pm lm}^\dagger L_i \right) \\ &\quad \times \left(\sigma_i \Omega_{q'_\pm l'm'} + \frac{1}{\kappa'_\pm} L_i \Omega_{q'_\pm l'm'} \right) \\ &\to 2\eta^2 \int d\Omega\, \partial_\mu \xi^\dagger(\theta,\phi) \partial^\mu \xi(\theta,\phi) \end{aligned} \tag{6.2.52}$$

This term describes the propagation of the T-spinor on the 4-plane. In summary, the effective six-dimensional bosonic action of the Higgsed $\mathcal{N} = 1^*$ theory is,

$$\begin{aligned} \mathcal{S}_B = \frac{N}{4\pi g_{ym}^2} \eta^2 \int d^4x \int d\Omega \Big(&- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2\partial_\mu n_a \partial^\mu n^a - 2\partial_\mu \xi^\dagger \partial^\mu \xi \\ &+ \frac{1}{2} A_\mu \Delta_{S^2} A^\mu - \mathcal{F}_{ab} \mathcal{F}^{ab} - 2\,\xi^\dagger \kappa^2 \xi \Big) \end{aligned} \tag{6.2.53}$$

By comparing this action with the bosonic action of the Maldacena-Núñez compactified gauge theory, a classical correspondence can be observed. Recall that T-spinors of the Maldacena-Núñez action are in the spherical basis, whilst the T-spinors of the action above are in the cartesian basis. In order to faciliate a comparison between the actions, the T-spinors must all be in the same basis. A conversion between the T-spinors bases is performed by the transformation of the T-spinors $\xi = V^\dagger \Xi$ and the similarity transformation of the Dirac operator (3.4.62). Furthermore, the fields should have their canonical mass dimension. The mass parameter η should be absorbed into the fields.[5]

$$\begin{aligned}\mathcal{S}_B = \frac{1}{g_{ym}^2}\frac{N}{4\pi R^2}\int d^4x \int R^2 d\Omega\Big(&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\partial_\mu n_a \partial^\mu n^a - 2\partial_\mu \Xi^\dagger \partial^\mu \Xi \\ &+ \frac{1}{2}A_\mu \Delta_{S^2} A^\mu - \mathcal{F}_{ab}\mathcal{F}^{ab} - 2\,\Xi^\dagger(-i\hat{\nabla}_{S^2})^2 \Xi\Big)\end{aligned} \tag{6.2.54}$$

With appropriate re-scaling of the fields, the action above is identical to the bosonic action of the Maldacena-Núñez compactified gauge theory (4.2.12) with the identification,

$$g_6^2 = \frac{4\pi R^2}{N}\, g_{ym}^2 \tag{6.2.55}$$

The treatment of the fermions is less obvious. The Yukawa potential of the Higgsed $\mathcal{N} = 1^*$ theory describes interactions between the gaugino, the chiral fermions and the complex scalars.

$$\mathcal{L}_y = \eta^4 \int d\Omega \left\{ i\bar{\Psi}_i \varepsilon_{ijk} L_k \Psi_j + 2i\bar{\Psi}_i L_i \Lambda - \bar{\Psi}_i \Psi_i \right\} \tag{6.2.56}$$

The bosonic 2-component object $\mathcal{Y}_i$ must have a fermionic partner which is related via a supersymmetry transformation. Under a supersymmetry transformation the complex scalars transform as,

$$\delta_\epsilon \delta \Phi_i = \epsilon \psi_i, \qquad \delta_\epsilon \delta \Phi_i^\dagger = \bar{\epsilon}\bar{\psi}_i \tag{6.2.57}$$

The corresponding supersymmetry transformation of $\mathcal{Y}_i$ is,

$$\delta_\epsilon \mathcal{Y}_i = \frac{1}{\sqrt{2}}\begin{pmatrix} \delta_\epsilon a_i \\ \delta_\epsilon (i b_i) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon\psi_i + \bar{\epsilon}\bar{\psi}_i \\ \epsilon\psi_i - \bar{\epsilon}\bar{\psi}_i \end{pmatrix} \tag{6.2.58}$$

[5]The fields A_μ, n_a and ξ are all four-dimensional bosons, so their canonical mass dimension is one.

For the Majorana spinor,

$$E = \begin{pmatrix} \epsilon \\ \bar{\epsilon} \end{pmatrix} \tag{6.2.59}$$

The components of equation (6.2.58) are given by,

$$\epsilon\psi_i + \bar{\epsilon}\bar{\psi}_i = \bar{E}\Psi_i \qquad \epsilon\psi_i - \bar{\epsilon}\bar{\psi}_i = \bar{E}i\gamma_5\Psi_i \tag{6.2.60}$$

In terms of Majorana spinors, the supersymmetry transformation of $\mathcal{Y}_i$ is,

$$\delta_\epsilon \mathcal{Y}_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{E}\Psi_i \\ i\bar{E}\gamma_5\Psi_i \end{pmatrix} \tag{6.2.61}$$

This suggests that the fermionic counterpart of $\mathcal{Y}_i$ is,

$$\mathcal{X}_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_i \\ i\gamma_5\Psi_i \end{pmatrix} \tag{6.2.62}$$

With the fermionic superpartner of the bosonic 2-component object $\mathcal{Y}_i$ identified the Yukawa potential can be rewritten. However the Yukawa potential also contains a gaugino which must have an associated a 2-component object. Define the two fermionic 2-component objects,

$$\mathcal{X}^{\hat{\alpha}}_{iA} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{iA} \\ i\left(\gamma_5\right)_A{}^B \Psi_{iB} \end{pmatrix} \qquad \mathcal{Z}^{\hat{\alpha}}_A = \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda_A \\ i\left(\gamma_5\right)_A{}^B \Lambda_B \end{pmatrix} \tag{6.2.63}$$

where A is the $SO(3,1)$ spinor index for the Majorana spinors. In terms of these 2-component objects the Yukawa potential is,

$$\mathcal{L}_y = \eta^4 \int d\Omega \left\{ \bar{\mathcal{X}}^A_{i\hat{\alpha}} (\hat{\Delta}_{ij})^{\hat{\alpha}B}_{\hat{\beta}A} \mathcal{X}^{\hat{\beta}}_{jB} + 2i\, \bar{\mathcal{X}}^A_{i\hat{\alpha}} \left(\delta^{\hat{\alpha}}{}_{\hat{\beta}} \delta_A{}^B L_i \right) \mathcal{Z}^{\hat{\beta}}_B \right\} \tag{6.2.64}$$

with $\hat{\Delta}_{ij} = \delta^{\hat{\alpha}}{}_{\hat{\beta}} \delta_A{}^B \left(i\varepsilon_{ijk} L_k - \delta_{ij} \right)$. The 2-component object $\mathcal{X}^{\hat{\alpha}}_{iA}$ is expanded in the complete basis of eigenstates of the operator $\hat{O}_{ij}$. The same expansion as the bosons is consistent as the bosons and fermions have the same mass squared matrix and hence the fields are all transformed under the same similarity matrix.

$$\mathcal{X}^{\hat{\alpha}}_{iA} = \mathcal{B}^{\hat{\alpha}}_{iA} + \mathcal{R}^{\hat{\alpha}}_{iA} \tag{6.2.65}$$

with,

$$\mathcal{B}^{\hat{\alpha}}_{iA} = \sum_{lm} u^{\hat{\alpha}}_{lmA} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm}(\theta,\phi) \tag{6.2.66a}$$

$$\mathcal{R}^{\hat{\alpha}}_{iA} = \sum_{lm}\sum_{+,-} \zeta^{(\pm)}_{lmA} \left((\sigma_i)^{\hat{\alpha}}{}_{\hat{\beta}} \Omega^{\hat{\beta}}_{q_\pm lm}(\theta,\phi) + \frac{1}{\kappa_\pm} L_i \Omega^{\hat{\alpha}}_{q_\pm lm}(\theta,\phi) \right) \tag{6.2.66b}$$

where $\zeta^{(\pm)}_{lmA}$ is a $SO(3,1)$ spinor coefficient. The coefficient $u^{\hat{\alpha}}_{lmA}$ is a $SO(3,1)$ spinor for each $\{l,m\}$ and is a 2-component object like its bosonic counterpart $v^{\hat{\alpha}}_{lm}$.

$$u^{\hat{\alpha}}_{lmA} = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta_A{}^B \\ i\,(\gamma_5)_A{}^B \end{pmatrix} u_{lmB} \tag{6.2.67}$$

The 2-component object $\mathcal{Z}_A$ is related to the gaugino and is expanded in spherical harmonics,

$$\mathcal{Z}^{\hat{\alpha}}_A(\theta,\phi) = \sum_{lm} \mathcal{Z}^{\hat{\alpha}}_{lmA} Y_{lm}(\theta,\phi) \tag{6.2.68}$$

where $\mathcal{Z}^{\hat{\alpha}}_{lmA}$ is a $SO(3,1)$ spinor coefficient and an arbitrary 2-component object like $u^{\hat{\alpha}}_{lmA}$.

$$\mathcal{Z}^{\hat{\alpha}}_{lmA} = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta_A{}^B \\ i\,(\gamma_5)_A{}^B \end{pmatrix} \Lambda_{lmB} \tag{6.2.69}$$

The fermionic calculations follow the same procedure as the bosonic calculations. Under the expansion of the fermions in the complete basis of eigenstates the following terms of the Yukawa potential become,

$$\int d\Omega \left\{ i\bar{\mathcal{X}}_i \varepsilon_{ijk} L_k \mathcal{X}_j - \bar{\mathcal{X}}_i \mathcal{X}_i \right\} = \int d\Omega \Big\{ \bar{\mathcal{B}}_i \hat{\Delta}_{ij} \mathcal{B}_j + \bar{\mathcal{B}}_i \hat{\Delta}_{ij} \mathcal{R}_j + \bar{\mathcal{R}}_i \hat{\Delta}_{ij} \mathcal{B}_j + \bar{\mathcal{R}}_i \hat{\Delta}_{ij} \mathcal{R}_j \Big\}$$

The first three terms are zero because the following expression is zero.

$$i\varepsilon_{ijk} L_k \mathcal{B}^{\hat{\alpha}}_{jA} - \mathcal{B}^{\hat{\alpha}}_{iA} = 0 \tag{6.2.70}$$

The only non-zero term is,

$$\eta^4 \int d\Omega\, \bar{\mathcal{R}}_i \hat{\Delta}_{ij} \mathcal{R}_j \rightarrow \frac{1}{2}\eta^3 \int d\Omega\, \bar{\zeta}^A_{\hat{\alpha}}(\theta,\phi)\, \kappa^{\hat{\alpha}}{}_{\hat{\beta}}\, \zeta^{\hat{\beta}}_A(\theta,\phi) \tag{6.2.71}$$

where $\zeta(\theta,\phi)$ is a fermionic T-spinor,

$$\zeta^{\hat{\alpha}}_A(\theta,\phi) = \sum_{lm}\sum_{+,-} \zeta^{(\pm)}_{lmA}\Omega^{\hat{\alpha}}_{q\pm lm} \tag{6.2.72}$$

This term describes the propagation of the fermionic T-spinor on the 2-sphere. The remaining term of the Yukawa potential is,

$$2\eta^4 \int d\Omega\, i\bar{\Psi}_i L_i \Lambda = 2\eta^4 \int d\Omega \left\{ i\bar{\mathcal{B}}^{\hat{\alpha}}_{iA} L_i \mathcal{Z}^A_{\hat{\alpha}} + i\bar{\mathcal{R}}^{\hat{\alpha}}_{iA} L_i \mathcal{Z}^A_{\hat{\alpha}} \right\} \tag{6.2.73}$$

The second term of equation (6.2.73) vanishes, but the first term is,

$$2\eta^4 \int d\Omega\, i\bar{\mathcal{B}}^{\hat{\alpha}}_{iA} L_i \mathcal{Z}^A_{\hat{\alpha}} = \eta^3 \int d\Omega \frac{1}{\sqrt{g}} \bar{\mathcal{G}}_{\theta\phi A}\Lambda^A = \frac{1}{2}\eta^3 \int d\Omega \frac{1}{\sqrt{g}} \varepsilon^{ab}\bar{\mathcal{G}}_{abA}\Lambda^A \tag{6.2.74}$$

The 'field tensor' of the fermionic T-vector $g_a(\theta,\phi)$ is $\mathcal{G}_{ab} = \partial_a g_b - \partial_b g_a$ and $\varepsilon^{\theta\phi} = 1$. The fermionic T-vector g_a is analogous to the bosonic T-vector n_a,

$$g_{\theta A}(\theta,\phi) = R\sum_{lm} u_{lmA} \frac{(-1)}{\sqrt{l(l+1)}} \csc\theta\, \partial_\phi Y_{lm}(\theta,\phi) \tag{6.2.75a}$$

$$g_{\phi A}(\theta,\phi) = R\sum_{lm} u_{lmA} \frac{1}{\sqrt{l(l+1)}} \sin\theta\, \partial_\theta Y_{lm}(\theta,\phi) \tag{6.2.75b}$$

$$\frac{1}{\sqrt{g}}\mathcal{G}_{\theta\phi A} = R\sum_{lm} u_{lmA} \frac{1}{\sqrt{l(l+1)}} \Delta_{S^2} Y_{lm}(\theta,\phi) \tag{6.2.75c}$$

The gaugino has a Fourier expansion in spherical harmonics.

$$\Lambda_A(\theta,\phi) = \sum_{lm} \Lambda_{lmA} Y_{lm}(\theta,\phi) \tag{6.2.76}$$

In summary, the Yukawa potential has become,

$$\mathcal{S}_y = \frac{N}{4\pi g^2_{ym}}\eta^3 \int d^4x \int d\Omega \left\{ \frac{1}{2}\bar{\zeta}\,\kappa\,\zeta + \frac{1}{2}\frac{1}{\sqrt{g}}\varepsilon^{ab}\bar{\mathcal{G}}_{ab}\Lambda \right\} \tag{6.2.77}$$

The second term in the Yukawa potential is very interesting. It is a term linear in derivatives that couples the fermionic T-scalar to the fermionic T-vector. It appears to be a kinetic terms for both the fermionic T-scalars and fermionic T-vectors on the 2-sphere.

The remaining fermionic terms are the kinetic terms. The kinetic term of the gauginos is trivial under the Fourier expansion. As there are no chiral fermions in this term it is does not need to be rewritten in terms of $\mathcal{Z}$. As its Fourier expansion is in terms of spherical harmonics, the kinetic term of the gaugino is unchanged under a Fourier expansion and continues to describe the propagation of the gaugino (fermionic T-scalar) on the 4-plane.

$$\eta^3 \int d\Omega \, \frac{i}{2} \bar{\Lambda} \gamma^\mu \partial_\mu \Lambda \tag{6.2.78}$$

The kinetic term of the chiral spinors must be rewritten in terms of $\mathcal{X}_i$.

$$\eta^3 \int d\Omega \, \bar{\Psi}_i^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \Psi_{iB} = \eta^3 \int d\Omega \, \bar{\mathcal{X}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{X}_{iB}^{\hat{\alpha}}$$

Under the Fourier expansion of $\mathcal{X}_i$,

$$\eta^3 \int d\Omega \, \bar{\mathcal{X}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{X}_{iB}^{\hat{\alpha}} = \eta^3 \int d\Omega \Big\{ \bar{\mathcal{B}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{B}_{iB}^{\hat{\alpha}} + \bar{\mathcal{R}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{R}_{iB}^{\hat{\alpha}} \Big\} \tag{6.2.79}$$

The first term in this expansion is,

$$\begin{aligned} &\eta^3 \int d\Omega \, \bar{\mathcal{B}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{B}_{iB}^{\hat{\alpha}} \\ &\quad = \eta^3 \int d\Omega \sum_{lm} \sum_{l'm'} \left(\bar{u}_{lm}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu u_{l'm'B} \right) \frac{1}{\sqrt{l(l+1)l'(l'+1)}} Y_{lm}^\dagger L^2 Y_{l'm'} \end{aligned} \tag{6.2.80}$$

In analogy with the bosons the fermionic T-vector can be written as,

$$\sum_{lm} u_{lmA} \frac{1}{\sqrt{l(l+1)}} i L_i Y_{lm} = \frac{1}{R} \frac{1}{\sqrt{g}} k_i^a g_{ab} \varepsilon^{bc} g_{cA} \tag{6.2.81}$$

and therefore the term describes the propagation of the fermionic T-vector on the 4-plane.

$$\eta^3 \int d\Omega \, \bar{\mathcal{B}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{B}_{iB}^{\hat{\alpha}} = \eta^3 \int d\Omega \, \bar{g}_a^A \left(\gamma^\mu\right)_A{}^B \partial_\mu g_B^a \tag{6.2.82}$$

The second term of the Fourier expansion (6.2.79) is,

$$\eta^3 \int d\Omega \, \bar{\mathcal{R}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{R}_{iB}^{\hat{\alpha}} \to \eta^3 \int d\Omega \, \bar{\zeta}_{\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \zeta_B^{\hat{\alpha}} \tag{6.2.83}$$

and describes the propagation of the fermionic T-spinor on the 4-plane.

In summary, the full six-dimensional effective action of the Higgsed $\mathcal{N} = 1^*$ theory is (contracting all spinor indices),

$$\begin{aligned} \mathcal{S} = \frac{N}{4\pi g_{ym}^2}\eta^2 \int d^4x \int d\Omega \Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\eta\bar{\Lambda}\gamma^\mu\partial_\mu\Lambda - \frac{i}{2}\eta\bar{g}_a\gamma^\mu\partial_\mu g^a \\ & - \frac{i}{2}\eta\bar{\zeta}\gamma^\mu\partial_\mu\zeta - 2\partial_\mu n_a\partial^\mu n^a - 2\partial_\mu\xi^\dagger\partial^\mu\xi + \frac{1}{2}A_\mu\Delta_{S^2}A^\mu \\ & - \mathcal{F}_{ab}\mathcal{F}^{ab} - 2\,\xi^\dagger\kappa^2\xi + \frac{1}{2}\eta\,\bar{\zeta}\,\kappa\,\zeta + \frac{1}{2}\frac{1}{\sqrt{g}}\eta\,\varepsilon^{ab}\bar{\mathcal{G}}_{ab}\Lambda \Big\} \end{aligned} \tag{6.2.84}$$

As with the bosonic action (6.2.53) all the T-spinor fields are expressed in terms of the cartesian basis of spherical spinors. It is important to see the effective action in terms of T-spinors fields in the spherical basis of spherical spinors. The transformation between bases was illustrated for the bosonic action. The fields should also be given their canonical mass dimension. Applying the transformation to the full action,

$$\begin{aligned} \mathcal{S} = \frac{1}{g_{ym}^2}\frac{N}{4\pi R^2} \int d^4x \int R^2\, d\Omega \Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\bar{\Lambda}\gamma^\mu\partial_\mu\Lambda - \frac{i}{2}\bar{g}_a\gamma^\mu\partial_\mu g^a \\ & - \frac{i}{2}\bar{\Upsilon}\gamma^\mu\partial_\mu\Upsilon - 2\partial_\mu n_a\partial^\mu n^a - 2\partial_\mu\Xi^\dagger\partial^\mu\Xi + \frac{1}{2}A_\mu\Delta_{S^2}A^\mu \\ & - \mathcal{F}_{ab}\mathcal{F}^{ab} - 2\,\Xi^\dagger(-i\hat{\nabla}_{S^2})^2\,\Xi + \frac{1}{2}\bar{\Upsilon}\hat{\gamma}_3(-i\hat{\nabla}_{S^2})\Upsilon + \frac{1}{2}\frac{1}{\sqrt{g}}\varepsilon^{ab}\bar{\mathcal{G}}_{ab}\Lambda \Big\} \end{aligned} \tag{6.2.85}$$

This effective six-dimensional action is classically equivalent to the action of the Maldacena-Núñez compactified gauge theory. This equivalence can be seen through a direct comparison of the bosonic parts of the two actions. As the two theories are supersymmetric this comparison is sufficient to show that the full actions are classically equivalent. The fields of the theory are summarized below.

Fields	T-Spin	Spin
A_μ	T-scalar	
Ξ	T-spinor	Bosons
n_a	T-vector	
Λ	T-scalar	
Υ	T-spinor	Fermions
g_a	T-vector	

The action of the Maldacena-Núñez compactified gauge theory has some unusual features that deserve comment. The unusual features are due to the topological twist used to preserve supersymmetry when compactifying the six-dimensional gauge theory and are apparent in the kinetic terms for propagation on the 2-sphere. The bosonic T-scalars, fermionic T-spinors and bosonic T-vectors are all realised as scalar fields, spinor fields and gauge fields on a 2-sphere, respectively. The kinetic term of the bosonic T-spinors on the 2-sphere is unusual as it is realised as the square of the Dirac operator. In fact this realisation makes perfect sense. The kinetic term of a boson is quadratic in derivatives whilst a fermion is linear in derivatives. It is not possible to remove a derivative from the action, therefore the bosonic T-spinors must be quadratic in derivatives. In a standard field theory, the kinetic term for spinors fields is the Dirac operator, which is linear in derivatives. It is sensible for the kinetic term of the bosonic T-spinor to be the square of the Dirac operator on the 2-sphere. It is found that the fermionic T-scalars and fermionic T-vectors have a coupled kinetic term. It appears to be some combined square root of a scalar Laplacian and a Maxwell term. This also makes sense for the same reason as the bosonic T-spinor. The fields are fermions and hence linear in derivatives, so the kinetic term for the fermionic T-scalars and T-vectors must be a "square root" of the canonical kinetic term for scalars and vectors, respectively. This can only be achieved through this coupling of the T-scalars and T-vectors, which is also suggested by the classical Kaluza-Klein spectrum. In order to form the Maldacena-Núñez fields into $\mathcal{N} = 1$ multiplets, the T-scalars had to be combined with the T-vectors to obtain massive vector multiplets.

Chapter 7

Concluding Remarks

The Maldacena-Núñez background provides an important tool in the study of confining gauge theories in four spacetime dimensions. It is a step towards a gravity dual of $\mathcal{N} = 1$ SUSY Yang-Mills in four dimensions. The six-dimensional nature of the dual gauge theory does restrict the scope of the duality; the inability of the supergravity approximation to decouple the Kaluza-Klein modes of the 2-sphere prevents this tool from studying a purely four-dimensional confining gauge theory. This Thesis used deconstruction to identify the purely four-dimensional gauge theory dual to the Maldacena-Núñez supergravity background. Deconstruction interpretes the Maldacena-Núñez compactified dual gauge theory as a limit of the four-dimensional $\mathcal{N} = 1^*$ theory. In particular, it is the $N \to \infty$ limit of the Higgs phase. A classical equivalence between the two theories has been demonstrated through the direct comparision of the classical spectra and actions. Furthermore, it can be argued that this equivalence persists at the quantum level between the $\mathcal{N} = 1^*$ theory and the Maldacena-Núñez compactified LST, the UV completion of the Maldacena-Núñez compactified gauge theory. The equivalence between the Maldacena-Núñez compactified LST and the $\mathcal{N} = 1^*$ theory shows that the $\mathcal{N} = 1^*$ theory is the dual gauge theory to the full string solution of the Maldacena-Núñez background.

An obvious extension to this Thesis is to study the deconstruction of the Maldacena-Núñez compactified LST in analogy to the study performed by Dorey of the toroidally compactified LST [20]. Furthermore, the equivalence identified in this Thesis, demonstrates that somehow the theory on the worldvolume of D5-branes compactified on a non-trivial 2-cycle of a CY_3 is equivalent to the theory on the worldvolume of D5-branes compactified on a trivial 2-cycle in the presence of an external flux, in the limit $N \to \infty$. It would be interesting to study the transition between these two D-brane pictures as $N \to \infty$.

The are many opportunities in which to apply the M(atrix) theory approach of deconstruction to other AdS/CFT dualities involving both D-branes probing orbifolds (as in [19, 20]) and D-branes wrapping Calabi-Yau manifolds. A particularly interesting theory is the β-deformed $\mathcal{N} = 1^*$ SUSY Yang-Mills theory. It has the superpotential,

$$\mathcal{W}(\Phi_i) = \mathrm{Tr}\left[e^{\frac{i\beta}{2}}\Phi_1\Phi_2\Phi_3 - e^{-\frac{i\beta}{2}}\Phi_1\Phi_3\Phi_2 + \eta\Phi_1^2 + \eta\Phi_2^2 + \eta\Phi_3^2\right]$$

and represents an intermediate step between the β-deformed $\mathcal{N} = 4$ theory studied by Dorey [20] and the $\mathcal{N} = 1^*$ theory studied in this Thesis. The additional dimensions that would emerge in the Higgs vacuum of this theory would be some higher-genus manifold. As an intermediate step between the toroidal and spherical cases already studied, it provides a platform in which to study the transition between the six-dimensional gauge/little string theories and the transition between their gravity duals.

Appendix A

Conventions and Spinor Identities

This Thesis uses the conventions of Wess and Bagger [27] with the metric signature $\eta_{\mu\nu} = (-,+,+,+,\dots)$. The supersymmetry algebra of Section 2 was presented in terms of $SO(3,1)$ Weyl spinors. The treatment of spinors in diverse dimensions was presented in Section 3.1. This Appendix serves to complement this Section by providing notational reference and further information on the spinor representations used in this Thesis.

The defining representation of the Lorentz group $SO(3,1)$ presented in Section 2 is,

$$M_{\mu\nu} = -\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \tag{A.1}$$

The indices μ, ν denote four-dimensional spacetime coordinates. The Lie algebra for this group was presented in equation (2.1.2a). The generators can be split into generators of rotations J_i (the $SO(3)$ subgroup of $SO(3,1)$) and boosts K_i [52].

$$\begin{aligned} J_i &= \tfrac{1}{2} i\varepsilon_{ijk} M_{jk} \\ K_i &= iM_{0i} \end{aligned} \qquad i,j,k = 1,2,3$$

By defining the following linear combinations of these two generators,

$$N_i^{(\pm)} = \frac{1}{2}\left(J_i \pm iK_i\right) \tag{A.2}$$

the Lie algebra decomposes into two distinct $SU(2)$ subalgebras. There is a direct relationship between $SO(3,1)$ and $SU(2)\times SU(2)$. They are not locally isomorphic, the Lie group $SO(4)$ is locally isomorphic to $SU(2) \times SU(2)$ and it is also locally isomorphic to $SO(3,1)$. It is this relationship which allows $SO(3,1)$ to be expressed in terms of $SU(2) \times SU(2)$ representations. In fact $SO(3,1)$ is locally isomorphic to the group $SL(2,\mathbb{C})$ due to the factor of i in the linear combination of generators [52].

The relationship between $SO(3,1)$ and $SL(2,\mathbb{C})$ of complex 2×2 matrices can be seen by considering the 2×2 matrix,

$$P = P_\mu \sigma^\mu = \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix} \tag{A.3}$$

where P_μ is an $SO(3,1)$ vector and $\sigma^\mu = (-\mathbb{1}, \vec{\sigma})$. The matrices $\vec{\sigma}$ are the Pauli sigma matrices.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.4}$$

The determinant of the matrix P is $\det P = P^2$. Consider a transformation of the matrix P.

$$\begin{aligned} P' &= APA^\dagger \\ \sigma^\mu P'_\mu &= A\sigma^\mu P_\mu A^\dagger \end{aligned} \tag{A.5}$$

If the transformation matrix A has determinant $\det A = 1$ then,

$$\det P' = \det A \det P \det A^\dagger = \det P \tag{A.6}$$

The matrix A is an element of $SL(2,\mathbb{C})$ and induces a Lorentz transformation on the vector P_μ as $\det P = P^2$ is invariant under this transformation.

The 2-component Weyl spinors of $SO(3,1)$ transform under the group $SL(2,\mathbb{C})$ [27].

$$\begin{aligned} \psi'_\alpha &= A_\alpha{}^\beta \psi_\beta \qquad & \psi'^\alpha &= A^{-1}{}_\alpha{}^\beta \psi^\beta \\ \bar{\psi}'_{\dot\alpha} &= A^*{}_{\dot\alpha}{}^{\dot\beta} \bar{\psi}_{\dot\beta} \qquad & \bar{\psi}'^{\dot\alpha} &= (A^*)^{-1}{}_{\dot\alpha}{}^{\dot\beta} \psi^{\dot\beta} \end{aligned} \tag{A.7}$$

Following from the Van der Waerden notation of dotted and undotted indices, the sigma matrix has the indices,

$$(\sigma^\mu)_{\alpha\dot\alpha},$$

The spinor indices are raised and lowered by the objects,

$$\varepsilon^{\alpha\beta} = i\sigma^2 \qquad \varepsilon_{\alpha\beta} = -i\sigma^2 \tag{A.8}$$

Spinors of the Lorentz group $SO(3,1)$ are 4-component objects. The Gamma matrices of the $SO(3,1)$ Clifford algebra are,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A.9}$$

where $\sigma = \{-\mathbb{1}, \vec{\sigma}\}$, $\bar{\sigma} = \{-\mathbb{1}, -\vec{\sigma}\}$. From the Clifford algebra the spinor representation of the Lorentz group is,

$$M^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \tag{A.10}$$

where the generators of the group $SL(2,\mathbb{C})$ are,

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha{}^\beta &= \frac{1}{4}\Big((\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\beta} - (\sigma^\nu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\Big) \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &= \frac{1}{4}\Big((\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}(\sigma^\nu)_{\alpha\dot{\beta}} - (\bar{\sigma}^\nu)^{\dot{\alpha}\alpha}(\sigma^\mu)_{\alpha\dot{\beta}}\Big) \end{aligned} \tag{A.11}$$

From the Clifford algebra the intertwiners can be calculated. For $SO(3,1)$,

$$A = \gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \tag{A.12}$$

The matrix that anticommutes with all the Gamma matrices is,

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{A.13}$$

From this matrix the chirality condition for the $SO(3,1)$ Lorentz group can be constructed.

$$\Psi_\pm = \frac{1}{2}\left(\mathbb{1}_4 \pm i\gamma^5\right)\Psi \tag{A.14}$$

The relationship between the 4-component spinors of $SO(3,1)$ and the 2-component Weyl spinors can be seen upon defining the 4-component spinor,

$$\Psi = \begin{pmatrix} \psi \\ \bar{\lambda} \end{pmatrix} \tag{A.15}$$

and applying the chirality condition. The left-handed chirality spinor Ψ_+ and the right-handed chirality spinor Ψ_- are,

$$\Psi_+ = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \qquad \Psi_- = \begin{pmatrix} 0 \\ \bar{\lambda} \end{pmatrix} \tag{A.16}$$

The Weyl spinors ψ and $\bar{\lambda}$ are the two non-zero components of the left-handed and right-handed chirality spinors, respectively. The Majorana condition for the $SO(3,1)$ Clifford algebra is,

$$\begin{pmatrix} \psi \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \psi^* \\ \bar{\lambda}^* \end{pmatrix} = \begin{pmatrix} -i\sigma^2\bar{\lambda}^* \\ i\sigma^2\psi^* \end{pmatrix} \tag{A.17}$$

This implies that,

$$\psi_\alpha = \varepsilon_{\alpha\beta}(\bar{\lambda}^{\dot{\beta}})^* \qquad \bar{\lambda}^\alpha = \varepsilon^{\dot{\alpha}\dot{\beta}}(\psi_\beta)^* \tag{A.18}$$

There are many spinor identities of the Clifford algebra [27]. For example, in 2-component form the Clifford algebra is,

$$\begin{aligned} (\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta &= -2\eta^{\mu\nu}\delta_\alpha{}^\beta \\ (\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} &= -2\eta^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{aligned} \tag{A.19}$$

Identities relevant to calculations in this Thesis are,

$$\begin{aligned} \operatorname{Tr}\sigma^\mu\bar{\sigma}^\nu &= -2\eta^{\mu\nu} \\ (\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} &= -2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}} \end{aligned} \tag{A.20}$$

These identities are used to convert between bispinors and 4-vectors of $SO(3,1)$ [27]. A bispinor is a direct product of a **(2,1)** and a **(1,2)** representation of $SO(3,1)$. It is defined as [27],

$$v_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}}v_\mu = \lambda_\alpha\bar{\lambda}_{\dot{\alpha}} \tag{A.21}$$

for a 4-vector v_μ. Conversely, the 4-vector is defined as,

$$v^\mu = -\frac{1}{2}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}v_{\alpha\dot{\alpha}} \tag{A.22}$$

Two further Clifford algebras are used in this Thesis, the Clifford algebra of $SO(5,1)$ is used to describe fermions on $\Re^{5,1}$ and together with the Clifford algebra of $SO(4)$ is used to describe fermions on $\Re^{9,1}$. The Clifford algebra for $SO(5,1)$ is [31],

$$\tilde{\Gamma}^i = \begin{pmatrix} 0 & \Sigma^i \\ \bar{\Sigma}^i & 0 \end{pmatrix} \tag{A.23}$$

where the components of the Clifford algebra are,

$$\begin{aligned} \Sigma^i &= \left(-i\eta^3, i\bar{\eta}^3, \eta^2, i\bar{\eta}^2, \eta^1, i\bar{\eta}^1\right) && \text{(A.24a)} \\ \bar{\Sigma}^i &= \left(i\eta^3, i\bar{\eta}^3, -\eta^2, i\bar{\eta}^2, -\eta^1, i\bar{\eta}^1\right) && \text{(A.24b)} \end{aligned}$$

The objects η^c and $\bar{\eta}^c$ are the 't Hooft eta symbols [31],

$$\bar{\eta}^c_{AB} = \eta^c_{AB} = \varepsilon_{cAB} \tag{A.25a}$$

$$\bar{\eta}^c_{4A} = \eta^c_{A4} = \delta_{cA} \tag{A.25b}$$

$$\eta^c_{AB} = -\eta^c_{BA} \qquad \bar{\eta}^c_{AB} = -\bar{\eta}^c_{BA} \tag{A.25c}$$

The intertwiners for $SO(5,1)$ are,

$$A = \tilde{\Gamma}_0 = \begin{pmatrix} 0 & \Sigma_0 \\ \bar{\Sigma}_0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & -\mathbb{1}_4 \\ -\mathbb{1}_4 & 0 \end{pmatrix} \tag{A.26}$$

The R-symmetry of the Maldacena-Núñez compactified gauge theory is $SO(4)$. The Clifford algebra for the $SO(4)$ R-symmetry is [31],

$$\tilde{\gamma}^m = \begin{pmatrix} 0 & \tau^m \\ \bar{\tau}^m & 0 \end{pmatrix} \tag{A.27}$$

with the components $\tau^m = (\vec{\sigma}, -i\mathbb{1})$ and $\bar{\tau}^m = (\vec{\sigma}, i\mathbb{1})$. The $SO(4)$ Clifford algebra has the following useful identities.

$$\begin{aligned} \mathrm{Tr}\, \tau^m \bar{\tau}^n &= 2\,\delta^{mn} && \text{(A.28a)} \\ (\tau^m)_{\underline{\alpha}\underline{\dot{\alpha}}} (\bar{\tau}_m)^{\underline{\dot{\beta}}\underline{\beta}} &= 2\,\delta_{\underline{\alpha}}{}^{\underline{\beta}}\delta_{\underline{\dot{\alpha}}}{}^{\underline{\dot{\beta}}} && \text{(A.28b)} \\ (\bar{\tau}^m)^{\underline{\alpha}\underline{\dot{\alpha}}} (\bar{\tau}_m)^{\underline{\dot{\beta}}\underline{\beta}} &= -2\,\varepsilon^{\underline{\alpha}\underline{\beta}}\varepsilon^{\underline{\dot{\alpha}}\underline{\dot{\beta}}} && \text{(A.28c)} \end{aligned}$$

An $SO(4)$ bispinor can be constructed in analogy to the $SO(3,1)$ bispinor.

$$\begin{aligned} v_{\underline{\alpha}\underline{\dot{\alpha}}} &= i(\tau^m)_{\underline{\alpha}\underline{\dot{\alpha}}}\phi_m \\ \phi^m &= -\frac{i}{2}(\bar{\tau}^m)^{\underline{\dot{\alpha}}\underline{\alpha}}v_{\underline{\alpha}\underline{\dot{\alpha}}} \end{aligned} \tag{A.29}$$

The bispinor is a representation of $SO(4)$ constructed from the direct product of two spinor representations of $SO(4)$. In this Thesis, the $SO(4)$ spinors are also representations of the group $SO(5,1)$.

$$v_{\underline{\alpha}}{}^{\underline{\dot{\alpha}}} = \lambda^A_{\underline{\alpha}}\bar{\lambda}^{\underline{\dot{\alpha}}}_A \tag{A.30}$$

Under the action of hermitian conjugation,

$$\begin{aligned} (v_{\underline{\alpha}}{}^{\underline{\dot{\alpha}}})^\dagger &= (\bar{\lambda}^{\underline{\dot{\alpha}}}_A)^\dagger(\lambda^A_{\underline{\alpha}})^\dagger = \Sigma^{0AB}\bar{\lambda}_{B\underline{\dot{\alpha}}}\bar{\Sigma}^0_{AC}\lambda^{C\underline{\alpha}} \\ &= \bar{\lambda}_{A\underline{\dot{\alpha}}}\lambda^{A\underline{\alpha}} = -\lambda^{A\underline{\alpha}}\bar{\lambda}_{A\underline{\dot{\alpha}}} = -v^{\underline{\alpha}}{}_{\underline{\dot{\alpha}}} \end{aligned} \tag{A.31}$$

Appendix B

Ordinary, Contravariant and Covariant Bases of a Vector Space

A spacetime manifold is an example of a vector space. A vector within a vector space is defined in terms of a vector (coordinate) basis. Let $\vec{a}_i$, $i = 1, \ldots, D$, be the D basis vectors of a D-dimensional vector space. A vector $\vec{V}$ within this vector space is defined in terms of the basis vectors,

$$\vec{V} = \sum_{i=1}^{D} V_i \, \vec{a}_i \tag{B.1}$$

The objects V_i are the components of the vector $\vec{V}$. The basis vectors are orthogonal and in the usual non-relativistic vector analysis they are orthonormal.

$$\vec{a}_i . \vec{a}_j = \delta_{ij} \tag{B.2}$$

For example, the unit vectors $\vec{\theta}$ and $\vec{\phi}$ form an orthonormal basis of the 2-sphere. In a relativistic theory there are two types of vector component for a vector $\vec{V}$, contravariant V^a and covariant V_a. The spacetime indices are raised and lowered by the spacetime metric.

$$V^a = g^{ab} \, V_b \qquad\qquad V_a = g_{ab} \, V^b \tag{B.3}$$

The contravariant and covariant basis vectors do not form an orthonormal basis [53]. Denote $\vec{e}^a$ as contravariant and $\vec{e}_a$ as covariant basis vectors on the 2-sphere. In a relativistic theory,

$$V^a V_a = V^2 \tag{B.4}$$

by definition. A vector field $\vec{V}$ can be expanded in covariant and contravariant components,

$$\vec{V} = V^a \vec{e}_a = V_a \vec{e}^a \tag{B.5}$$

Therefore,

$$V^2 = \vec{V}.\vec{V} = V^a \vec{e}_a V_b \vec{e}^b = V^a V_a \tag{B.6}$$

where the last equality is from the definition (B.4). In order for,

$$V^a \vec{e}_a V_b \vec{e}^b = V^a V_a \tag{B.7}$$

to be satisfied,

$$\vec{e}_a \vec{e}^b = \delta^b_a \tag{B.8}$$

the contravariant basis must be orthonormal to the covariant basis. Consequently, the contravariant basis,

$$\vec{e}^a \vec{e}^b = \vec{e}^a g^{bc} \vec{e}_c = g^{bc} \delta^a_b = g^{ac} \tag{B.9}$$

is not orthonormal. Similarly for the covariant basis.

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